Team Round Solutions

LMT Spring 2026

May 9, 2026

1. **[20]** Sam and Eddie love eating cakes. Sam eats 80% of his cake, while Eddie eats 52% of his cake. Interestingly, they ate the same amount of cake. If the size of Sam's cake was x% of the size of Eddie's cake before either of them ate their cakes, find x.

   Proposed by: Edwin Zhao

   Solution. $\boxed{65}$

   We have that the amount of cake Sam eats relative to Eddie's cake is $x\% \cdot 80\%$, and the amount of cake Eddie eats relative to his cake is 52%. So, $80x = 52$ and $\frac{52}{80} = 0.65$, so our answer is $\boxed{65}$. □

2. **[20]** Find the volume of the region that is swept out by a unit cube when it undergoes a 360° rotation about one of its edges.

   Proposed by: Peter Bai

   Solution. $\boxed{2\pi}$

   By sketching out the cube and imagining what it does when it rotates, our first observation will be that the region in question is a cylinder centered at the edge we selected to rotate about. Then, the lateral surface of the cylinder must have been swept out by the points in the cube that are furthest from the edge we selected. This is just the opposite edge, and we can use the Pythagorean theorem to find the distance between the two edges as $\sqrt{1^2+1^2} = \sqrt{2}$, which is also the radius of our cylinder.

   Finally, the volume of the cylinder is $\pi\left(\sqrt{2}\right)^2(1) = \boxed{2\pi}$. □

3. **[25]** Six perfectly logical people: Peter, William, Isabella, Jonathan, Jerry, and Ben, are playing a variation of Castlefall. Specifically, three distinct integers from 1 to 8 are chosen, and each integer is given to exactly two people, forming three teams of two. Each person only sees their own number, and everyone is aware of how the numbers were distributed. They have the following dialogue:

   - Peter: "My number is a multiple of 4."
   - William: "My number is prime."
   - Isabella: "My number is a factor of 5."
   - Jonathan: "My number is even."
   - Jerry: "My number is less than 5."
   - Ben: "My number is odd."
   - Peter: "I know who my partner is."
   - William: "I can identify the number each person has."

   Find the numbers that are given to each of Peter, William, Isabella, Jonathan, Jerry, and Ben as an ordered list of six integers in that order.

   Proposed by: James Wu

   Solution. $\boxed{(8,7,1,8,1,7)}$

   First of all, we can compute the sets

- Peter: $\{4, 8\}$
- William: $\{2, 3, 5, 7\}$
- Isabella: $\{1, 5\}$
- Jonathan: $\{2, 4, 6, 8\}$
- Jerry: $\{1, 2, 3, 4\}$
- Ben: $\{1, 3, 5, 7\}$

Since Peter knows his partner upon learning about the sets, notice that if his number is 4 he can pair with Jonathan and Jerry, which both can lead to valid pairings. Therefore, Peter has the number 8 and is on a team with Jonathan.

From here, we can perform casework on William's number:

- If William has 2 or 3, he can be paired with Jerry, but he wouldn't know what number Isabella and Ben have since it could be 1 or 5.
- If William has 5, he can be paired with Isabella or Ben, both of which work.
- If William has 7, he can only be paired with Ben, and Isabella and Jerry must have 1.

Therefore, from this information, only 7 works, so the numbers are

$$\boxed{(8, 7, 1, 8, 1, 7)}$$

for Peter, William, Isabella, Jonathan, Jerry, and Ben, respectively. □

4. **[30]** A number n satisfies the property that n is 36 times the sum of its digits. Find the sum of all possible values of n.

Proposed by: Ryan Tang

Solution. $\boxed{972}$

Note that $9 \mid n$, so 9 divides the sum of digits of n. Thus, $81 \mid n$, so $324 \mid n$. Now, suppose the largest solution has k digits. We have

$$n = 36 \cdot \text{the sum of digits} \leq 36 \cdot (9k) = 324k.$$

We also know $n \geq 10^{k-1}$. Thus, $10^{k-1} \leq 324k$. This is clearly false for $k \geq 5$. Thus, we only need to check 3 and 4 digit numbers. Suppose $n = 324\ell$. We have that the sum of digits is 9ℓ. Since n is at most 4 digits, $\ell \leq 4$. Manual testing gives the solutions are 324 and 648 which gives the answer of $\boxed{972}$. □

5. **[30]** Let a, b, c be nonzero real numbers such that

$$a = b^{2026} + c^{2026}$$
$$b = a^{2026} + c^{2026}$$
$$c = a^{2026} + b^{2026}.$$

Find the product of all possible values of a.

Proposed by: Henry Eriksson

Solution. $\boxed{2^{-1/2025}}$

Note first that all of a, b, c must be non-negative, as they are sums of even powers. Subtract the first equation from the second to receive

$$b - a = a^{2026} - b^{2026} = (a-b)\left(a^{2025} + a^{2024}b + \cdots + b^{2025}\right) \implies$$
$$(a-b)\left(1 + a^{2025} + a^{2024}b + \cdots + b^{2025}\right) = 0$$

The second expression is at least 1, as a and b are non-negative, so we must have $a - b = 0 \implies a = b$. By symmetry, $a = b = c$. Plugging that back into the equation, we receive

$$a = 2a^{2026} \implies a = \sqrt[2025]{\frac{1}{2}} = \left(\frac{1}{2}\right)^{2025}.$$

So, the only possibility for a is $\boxed{2^{-1/2025}}$. □

6. **[35]** Find the number of ways to replace each _ with either $+, -$, or $\times$ in the expression

$$1\ \_\ 2\ \_\ 3\ \_\ 4\ \_\ 5\ \_\ 6\ \_\ 7$$

such that the result evaluates to an odd number.

Proposed by: James Wu

Solution. $\boxed{364}$

First, we only need to consider the parities, which are

$$1, 0, 1, 0, 1, 0, 1$$

Addition and subtraction "split" the numbers into blocks, and we want the sum of the products of the blocks to be odd. For each 1, it can only contribute an odd number if it is not multiplied by a neighboring number. Since the neighboring gaps don't intersect, we can do a bit of casework. odd number is on the edge: there is one gap next to it which has 2 possibilities for not being $\times$ odd number isn't on the edge: there are two gaps next to it which gives $2 \times 2 = 4$ possibilities From here, we can even do more casework on the numbers we want to keep (which isn't that bad), or you can notice that the total number of ways is 3^6 and the difference between even and odd is

$$(1-2)(5-4)(5-4)(1-2) = 1$$

So the number of ways that result in an odd expression is

$$\frac{3^6-1}{2} = \boxed{364}$$

□

7. **[40]** Let ABC be a triangle with $AB = 6$. Let M be the midpoint of BC. A circle with its center on AC is tangent to AB at A and BC at M. The circle intersects AC at distinct points A and X. Find CX.

Proposed by: Samuel Tsui

Solution. $\boxed{2\sqrt{3}}$

Note that $AB = BM$ implies $BC = 2AB$. Furthermore, we can get $\angle BAC = 90^\circ$, so ABC is $30-60-90$ triangle. Thus $AC = 6\sqrt{3}$, and by Power of a Point, $CX = \frac{CM^2}{AC} = \boxed{2\sqrt{3}}$. □

8. **[40]** Let p, q, r be three primes, and define

$$k = \frac{pqr}{(p-2)(q-2)(r-2)}.$$

Find the sum of all possible integer values of k.

Proposed by: Ryan Tang

Solution. $\boxed{60}$

Let p be the smallest prime. Since $p-2 \nmid p, q, r$, it follows that $p = 3$.

- If all 3 of p, q, r are 3, then $k = 27$.
- If two of p, q, r are 3, then $r - 2 \mid 9$. Clearly, $r \in \{5, 11\}$. This gives $15, 11$.
- If there is only one 3, then $(q-2)(r-2) \mid 3qr$. Let q be the smaller of the two primes. Clearly, $(q-2, q) = (q-2, r) = 1$, so $q - 2 \mid 3$, so $q = 5$. Thus, $r - 2 \mid 5r$, so $r = 7$. This gives 7.

Thus, $27 + 15 + 11 + 7 = \boxed{60}$. □

9. **[40]** There are 2026 people at LMT, all of which go to Clarke, Diamond, or LHS.

- Before the individual rounds, every pair of people from the same school shake hands.

- Before the team round, every Clarke student shakes hands with every non-Clarke student.
- Before the guts round, every Clarke student shakes hands with every Diamond student.

Given that there were $2050312 = 2026 \times 1012$ hand shakes total, find the number of Clarke students at LMT.

Proposed by: Henry Eriksson

Solution. $\boxed{506}$

Let there be a Clarke students, b Diamond students, and c LHS students. We will double count the hand shakes by counting the number of hand shakes each student is a part of and summing over the students.

Each Clarke student shakes hands with every other student, and twice with every Diamond student, so they contribute $a(2025+b)$ hand shakes.

Each Diamond student shakes hands with every other Diamond student and every Clarke student twice, so they contribute $b(2a+b-1) = b(2025+a-c)$ hand shakes.

Each LHS student shakes hands with every other LHS student and with every Clarke student, so they contribute $c(a+c-1) = c(2025-b)$ hand shakes.

The total number of hand shakes counted twice is $2024 \cdot 2026$, so we have the equation

$$a(2025+b) + b(2025+a-c) + c(2025-b) = 2024 \cdot 2026 \implies$$

$$(2025a+ab) + (2025b+ab-bc) + (2025c-bc) = 2024 \cdot 2026 \implies$$

$$2025 * 2026 + 2ab - 2bc = 2024 \cdot 2026 \implies$$

$$bc - ab = 1013 \implies b(c-a) = 1013$$

Note that 1013 is prime and b is an integer. This means that $b = 1$ or $b = 1013$. If $b = 1$, then we have $a + c = 2025$ and $c - a = 1013$, so $\boxed{a = 506}$. If $b = 1013$, then we have $a + c = 1013$ and $c - a = 1$, so $\boxed{a = 506}$ also. □

10. **[45]** Define the function

$$f(t) = 20\left\lfloor \frac{t}{20} \right\rfloor - 26\left\lfloor \frac{t}{26} \right\rfloor.$$

Suppose that $f(t)$ has a minimum of m and a maximum of M. Find (m, M).

Proposed by: Ryan Tang

Solution. $\boxed{(-18, 24)}$

We make a few cosmetic rewrites for the purpose of a rigorous solution. First, note that $\left\lfloor \frac{x}{k} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor}{k} \right\rfloor$ for any integer k (and this works because $k \geq 1$ so $\frac{r}{k} < 1$ for $r < 1$). Thus,

$$\frac{1}{2} \cdot f(t) = 10\left\lfloor \frac{\frac{t}{2}}{10} \right\rfloor - 13\left\lfloor \frac{\frac{t}{2}}{13} \right\rfloor = 10\left\lfloor \frac{\lfloor \frac{t}{2} \rfloor}{10} \right\rfloor - 13\left\lfloor \frac{\lfloor \frac{t}{2} \rfloor}{13} \right\rfloor$$

so it suffices to maximize and minimize

$$g(t) = 10\left\lfloor \frac{t}{10} \right\rfloor - 13\left\lfloor \frac{t}{13} \right\rfloor$$

because $2g(\lfloor \frac{t}{2} \rfloor) = f(t)$.

Rewrite $g(t) = 13\left\{\frac{t}{13}\right\} - 10\left\{\frac{t}{10}\right\}$. Note that $0 \leq 13\left\{\frac{t}{13}\right\} < 13$ while $-10 < -10\left\{\frac{t}{10}\right\} \leq 0$. Adding the two, we have

$$-10 < g(t) < 13.$$

Thus, $-9 \leq g(t) \leq 12$. For the construction, we can use CRT. For example, if we want $g(t) = -9$, then pick $t \equiv -9 \pmod{10}, t \equiv 0 \pmod{13}$ (so $t \equiv 91 \pmod{130}$). For $g(t) = 12$, pick $t \equiv 0 \pmod{10}, t \equiv 12 \pmod{13}$ (so $t \equiv 120 \pmod{130}$). Thus, our answer is $\boxed{(-18, 24)}$. □

11. **[50]** Let a_n, b_n be sequences of numbers satisfying the recurrences

$$\frac{a_{n+1}}{a_n} = a_n^2 + 3b_n^2 \qquad \text{and} \qquad \frac{b_{n+1}}{b_n} = b_n^2 + 3a_n^2.$$

Given that $a_0 = 2$ and $b_0 = 1$, find a_{2026}.

Proposed by: Samuel Tsui

Solution. $\boxed{\dfrac{3^{3^{2026}}+1}{2}}$

Multiplying the equations by a_n and b_n respectively gives $a_{n+1} = a_n^3 + 3a_n b_n^2$ and $b_{n+1} = b_n^3 + 3b_n a_n^2$. Adding and subtracting these we have $a_{n+1} + b_{n+1} = (a_n + b_n)^3$ and $a_{n+1} - b_{n+1} = (a_n - b_n)^3$. Thus $a_{2026} + b_{2026} = 3^{3^{2026}}$ and $a_{2026} - b_{2026} = 1$ so $a_n = \boxed{\dfrac{3^{3^{2026}}+1}{2}}$. □

12. **[50]** Sam writes some of the numbers from 1 to 10 onto a chalkboard (at most one of each number and he writes at least one number). In one move, he chooses 2 distinct elements a and b currently on the chalkboard, erases them both, and writes down

$$\frac{\text{lcm}(a,b)}{\text{gcd}(a,b)}.$$

The process is repeated until one number remains on the board. Find the number of initial states such that there is a process in which the final number is a perfect square.

Proposed by: James Wu

Solution. $\boxed{63}$

Since for each prime p, lcm keeps the larger exponent and gcd keeps the smaller exponent, the quotient keeps the absolute difference between the exponents. Now, since the result needs to be a perfect square, we only care about the parity of the exponents. Since the parity of $|x-y|$ is the same as $x+y$ for all x and y, S would end with a perfect square if and only if the product of its elements is a perfect square.

From here, you can consider each prime factor's contribution.

The numbers $1,4,9$ are free since they don't contribute to the parity of the product.

Since 7 is the only number with the prime factor 7, it can't be chosen.

Only 5 and 10 contain the prime factor 5, so we must either choose both or neither of them. Similarly, we must choose either both or neither of 3 and 6.

Finally, there are exactly two ways to choose 2 and 8 to make the exponent of 2 in the product even.

Therefore, the answer is

$$2^3 \times 2 \times 2 \times 2 - 1 = \boxed{63}$$

□

13. **[55]** Let ABC be a triangle, let M be the midpoint of BC, and let G be the centroid of ABC. Suppose that the circumcircle of $\triangle BGM$ intersects AB at P and the circumcircle of $\triangle CGM$ intersects AC at Q. Given that $AP = \frac{1}{8}$, $AG = \frac{2}{11}$, and $AQ = \frac{1}{9}$, find the perimeter of quadrilateral $APGQ$.

Proposed by: Ryan Tang

Solution. $\boxed{\dfrac{17}{33}}$

Note that $AM = \frac{3}{2} \cdot AG = \frac{3}{11}$. We know that A, G, M, so A lies on the radical axis of the two circles. By POP, we have $AB = \frac{3}{2} \cdot \frac{AG^2}{AP} = \frac{48}{11^2}$ while $AC = \frac{3}{2} \cdot \frac{AG^2}{AQ} = \frac{54}{11^2}$. By parallelogram law, $BC^2 = 2AB^2 + 2AC^2 - 4AM^2 = \frac{78^2}{11^4}$, so $BC = \frac{78}{11^2}$. Thus, $BM = CM = \frac{39}{11^2}$. Hence, the perimeter is

$$AP + AQ + PG + GQ = (AP + AQ) + \frac{AP}{AM} \cdot BM + \frac{AQ}{AM} \cdot CM = (AP + AQ)\left(\frac{BM}{AM} + 1\right) = \frac{17}{72} \cdot \frac{24}{11} = \boxed{\frac{17}{33}}.$$

□

14. **[60]** Suppose that integers a, b, and c satisfy

$$10\left(a^2+b^2+c^2\right)+6\left(ab+bc+ca\right)=2^6\cdot 5\cdot 23.$$

Given that $0 \le a \le b \le c$, there exists a unique solution (a,b,c) to the equation. Find the value of $a+b+c$.

Proposed by: Peter Bai

Solution. $\boxed{34}$

Upon seeing this problem, we might notice that the equation could have been simplified by dividing both sides by 2. It turns out that this was an intentional decision on the part of the problem writer, as the left hand side admits the following equivalent form:

$$\begin{aligned}&10\left(a^2+b^2+c^2\right)+6(ab+bc+ca)\\&=\left(a^2+6ab+9b^2\right)+\left(b^2+6bc+9c^2\right)+\left(c^2+6ca+9a^2\right)\\&=(a+3b)^2+(b+3c)^2+(c+3a)^2.\end{aligned}$$

Since $a,b,c \ge 0$, each one of our three terms must be a nonnegative square number. This suggests that we should attempt to find a way to express $2^6\cdot 5\cdot 23$ as a sum of three squares, and, noticing that 2^6 is itself a square number, we can try to find a way to express $5\cdot 23 = 115$ as a sum of three squares first and then multiply by 2^6 at the end to get what we want. Indeed, after trying $115 = 10^2+n^2+m^2$ does not work, we immediately run into $115 = 9^2+5^2+3^2$. Thus, we must have

$$2^6\cdot 5\cdot 23 = 8^2\cdot 115 = 8^2\cdot 9^2+8^2\cdot 5^2+8^2\cdot 3^2 = 72^2+40^2+24^2.$$

The equation from the problem is now

$$(a+3b)^2+(b+3c)^2+(c+3a)^2 = 72^2+40^2+24^2,$$

which suggests setting up the following system of equations:

$$a+3b=72, \qquad b+3c=40, \qquad \text{and} \qquad c+3a=24.$$

Solving yields $a=6$, $b=22$, and $c=6$. Since the equation from the problem is symmetric, we can rearrange a, b, and c to get $(a,b,c)=(6,6,22)$, which satisfies our constraints. Our answer is then $6+6+22=\boxed{34}$.

Note: It turns out that orientation matters when setting up the system of equations. In particular, solving

$$3a+b=72, \qquad 3b+c=40, \qquad \text{and} \qquad 3c+a=24$$

gives solutions of $a=138/7$, $b=90/7$, *and* $c=10/7$, *which are not integers.* □

15. **[60]** In triangle ABC, D lie on the circumcircle and E lies on BC such that $AD \perp BC$ and $\angle BAE = \angle EAC$. Suppose $DE \perp AC$. Given that $AB=6$ and $AC=7$, find BC.

Proposed by: Ryan Tang

Solution. $\boxed{\dfrac{13}{\sqrt{7}}}$

Note that E is the orthocenter of ADC, so $\angle DAE = \angle BCD = \angle BAD$ so $AE = AB$. Now, suppose F is the midpoint of BE (or the foot of A to BC). Let $BF = 3x$, $AF = y$. By Angle-bisector theorem, $CE = 7x$. By pythagorean, we get the following two equations:

$$\begin{aligned}(3x)^2+y^2&=36\\(3x+7x)^2+y^2&=49\end{aligned}$$

Thus, $x^2 = \frac{13}{91} = \frac{1}{7}$. This gives the answer of $\frac{13}{\sqrt{7}}$. □