Speed Round Solutions

LMT Spring 2026

May 9, 2026

1. **[6]** Let $a \star b = \frac{a+b}{2}$. Find the value of

$$((1 \star 2) \star (3 \star 4)) \star ((5 \star 6) \star (7 \star 8)).$$

Proposed by: Samuel Tsui

Solution. $\boxed{\frac{9}{2}}$

Note every number is divided by 2 three times, so the answer is

$$\frac{1+2+\cdots+8}{8} = \boxed{\frac{9}{2}}.$$

□

2. **[6]** Let $MATH$ be a parallelogram with area 2026. Let L be the midpoint of MA. Find the area of LMT.

Proposed by: Samuel Tsui

Solution. $\boxed{\frac{1013}{2}}$

$$[LMT] = \frac{1}{2}[MAT] = \frac{1}{4}[MATH] = \boxed{\frac{1013}{2}}.$$

□

3. **[6]** Find the maximum possible value of $a^b + c - d$ where the ordered quadruple (a, b, c, d) is a permutation of $(2, 0, 2, 6)$.

Proposed by: Larry Cui

Solution. $\boxed{66}$

To maximize $a^b + c - d$, we wanted 2^6 to be included, which contributes to the value 64. We also further maximize this if $d = 0$, which leaves $c = 2$. So therefore, the maximum possible value is $\boxed{66}$. □

4. **[6]** Three standard fair 6-sided dice are rolled. Find the probability that the product of numbers on the top faces is prime.

Proposed by: James Wu

Solution. $\boxed{\frac{1}{24}}$

Two of the dice have to be 1 and the remaining one must be prime (either 2, 3, or 5). There are 3 ways to choose the dice that aren't 1, and 3 numbers to choose from, so the probability is

$$\frac{3 \cdot 3}{6^3} = \boxed{\frac{1}{24}}.$$

□

5. **[6]** Cup A contains a 1 : 5 mixture of sugar to water and cup B contains a 4 : 7 mixture of sugar to water. Cup C is a mixture of the solutions from Cup A and Cup B in a 2 : 3 ratio. Find the number of grams of sugar in 660 grams of Cup C solution.

   Proposed by: Larry Cui

   Solution. $\boxed{188}$

   $\frac{1}{6}$ of the solution of cup A is sugar and $\frac{4}{11}$ of the solution of Cup B is sugar. Note that $\frac{2}{5}$ of the solution of Cup C comes from Cup A and $\frac{3}{5}$ of the solution comes from Cup B. Then, the amount of sugar is $(\frac{1}{6})(\frac{2}{5}) + (\frac{4}{11})(\frac{3}{5}) = \frac{47}{165}$ of the solution in Cup C. Since we are sampling 660 grams of Cup C, the amount of sugar will be $\boxed{188}$. □

6. **[6]** Let ABC be an equilateral triangle with side length 1. Let D be the foot of the altitude from A to BC. The circle with diameter AD intersects AB and AC at points X and Y, respectively. Find XY.

   Proposed by: Samuel Tsui

   Solution. $\boxed{\frac{3}{4}}$

   By power of a point $BX = \frac{BD^2}{AB} = \frac{1}{4}$ so $XY = AX = AB - BX = \boxed{\frac{3}{4}}$. □

7. **[6]** Find the number of ways to arrange the numbers 1, 2, 3, 4, and 5 in a row such that the sum of any two adjacent numbers is not divisible by 3.

   Proposed by: James Wu

   Solution. $\boxed{8}$

   A number that is 1 mod 3 can't be next to a number that is 2 mod 3, while 3 can be next to anything since nothing else is 0 mod 3. Therefore, $\{1,4\}$ must be on one side of 3 and $\{2,5\}$ on the other side, so the number of ways is $2 \times 2 \times 2 = \boxed{8}$. □

8. **[6]** Let a, b, c be relatively prime positive integers satisfying

   $$a + b = c \qquad \text{and} \qquad 20a = 2b + 6c.$$

   Find the value of $a + b + c$.

   Proposed by: Samuel Tsui

   Solution. $\boxed{22}$

   Solving yields $7a = 4b$ and $11a = 4c$, so $a + b + c = 4 + 7 + 11 = \boxed{22}$. □

9. **[6]** Suppose primes p and q satisfy the property that $(p + 1)(q + 1)$ is not divisible by any perfect square greater than 2. Find the smallest possible value of $p + q$.

   Proposed by: Ryan Tang

   Solution. $\boxed{15}$

   If both are odd, then $4 \mid$ expression. Thus, WLOG $p = 2$. Then, $q = 3$ does not work, $q = 5$ does not work, $q = 7$ does not work, $q = 11$ does not work, $q = 13$ works. Thus, the answer is 15. □

10. **[6]** Rectangle $ABCD$ and triangle ABE share side AB and do not overlap. If $AB = 5, AE = 4, BE = 3$, and the area of $\triangle CDE$ is 21, find BC.

    Proposed by: Ryan Tang

    Solution. $\boxed{6}$

    Note that the area of $\triangle CDE$ is $\frac{1}{2}(BC + \frac{12}{5}) \cdot 5$ using the base times height formula. Thus, $BC = 6$. □

11. **[6]** Find the remainder when 10^{2026} is divided by 9801.

Proposed by: Ryan Tang

Solution. $\boxed{2278}$

Write $10^{2026} = (100)^{1013} = (1+99)^{1013} \equiv 1 + 1013 \cdot 99 + \cdots \equiv 1 + 1013 \cdot 99 \equiv 2278 \pmod{99^2}$. □

12. **[6]** Math team has 4 freshman, 4 sophomores, and 4 juniors. Find the number of ways there are to split them into 3 groups of 4 such that each group has at least one person from each grade.

Proposed by: Samuel Tsui

Solution. $\boxed{1728}$

Note that one team will have 2 freshman, one team will have 2 sophomores, and one team will have 2 juniors. Then after picking the pairs there are 2 students left in each grade: one per remaining team. Thus the answer is $\binom{4}{2}^3 \cdot 2!^3 = \boxed{1728}$. □

13. **[6]** Let f be a polynomial of degree 3 with leading coefficient 1. Suppose $f(2) = 67, f(-2) = 41$. Find $f(10) - f(-10)$.

Proposed by: Ryan Tang

Solution. $\boxed{2050}$

Suppose $f(x) = O(x) + E(x)$ where $O(x) = x^3 + bx$ is an odd function and $E(x) = ax^2 + c$ is an even function. Then, $f(x) - f(-x) = O(x) - O(-x) + E(x) - E(-x) = 2x^3 + 2bx$. Thus, $67 - 41 = f(2) - f(-2) = 16 + 4b$ so $b = \frac{5}{2}$. Thus, $f(10) - f(-10) = 2000 + 50 = \boxed{2050}$. □

14. **[6]** The numbers 1 through 9 are placed randomly in an 3×3 grid. Find the expected number of 2×2 subgrids which contain 4 consecutive numbers.

Proposed by: Samuel Tsui

Solution. $\boxed{\dfrac{4}{21}}$

For each of the four 2×2 subgrids there are $\binom{9}{4} = 126$ possible sets of numbers that can be placed inside the cells and 6 sets that have 4 consecutive numbers. Thus the answer is

$$4 \cdot \frac{6}{126} = \boxed{\frac{4}{21}}.$$

□

15. **[6]** The parabola $y = x^2$ is drawn. Two points A, B with integer coordinates are selected along the parabola on opposite sides of $x = 0$ such that the line passing through A and B has y-intercept 7. Find the sum of all possible y-coordinates of A.

Proposed by: Ryan Tang

Solution. $\boxed{50}$

Suppose A is on the left side of $x = 0$ and B is on the right. Suppose point A has y-coordinate a and point B has y-coordinate b. Since the points lie on the parabola and we know what half they are in, we can deduce that $A = (-\sqrt{a}, a)$ and $B = (\sqrt{b}, b)$. Now, the line through these points is given by $y - b = \frac{b-a}{\sqrt{b}+\sqrt{a}} \cdot (x - \sqrt{b})$ or $y - b = (\sqrt{b} - \sqrt{a})(x - \sqrt{b})$. To find the y-intercept, we plug in $x = 0$ to find $y = \sqrt{ab}$. We want $ab = 7^2$, so $a = 49, 1$. This gives $\boxed{50}$. □

16. **[6]** Find the area of a triangle with side lengths of $\sqrt{10001}$, $\sqrt{2501}$, and $\sqrt{2504}$.

Proposed by: Ryan Tang

Solution. $\boxed{75}$

Note that $10001 = 100^2 + 1^2, 2501 = 50^2 + 1^2, 2504 = 50^2 + 2^2$. Consider the trapezoid $ABCD$ such that $AB \perp BC$, $BC \perp CD$, $BC = 100$, $AB = 1$, and $CD = 2$. If M is the midpoint of BC, then we have that AMD is the triangle with those lengths. Thus, the area is

$$\frac{1+2}{2} \cdot 100 - \frac{1}{2} \cdot 50 - \frac{2}{2} \cdot 50 = \boxed{75}.$$

□

17. **[6]** Find the value of

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)n!}.$$

Proposed by: Samuel Tsui

Solution. $\boxed{\frac{1}{2}}$

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)n!} = \sum_{n=1}^{\infty} \frac{n+1}{(n+2)!} = \sum_{n=1}^{\infty} \frac{(n+2)-1}{(n+2)!} = \sum_{n=1}^{\infty} \frac{1}{(n+1)!} - \frac{1}{(n+2)!} = \frac{1}{(1+1)!} = \boxed{\frac{1}{2}}$$

□

18. **[6]** Find the minimum possible value of

$$4\left\lfloor \frac{b+c}{a} \right\rfloor + 3\left\lfloor \frac{c+a}{b} \right\rfloor + 4\left\lfloor \frac{a+b}{c} \right\rfloor,$$

where a, b, and c are the side lengths of a non-degenerate triangle.

Proposed by: Ryan Tang

Solution. $\boxed{14}$

By the triangle inequality, each of the floors must be at least 1. The main idea is that not all of them can be 1. Indeed, if they were all 1, then

$$b + c < 2a$$
$$c + a < 2b$$
$$a + b < 2c$$

but adding the 3 gives us a contradiction. Thus, the answer is $4 + 3 \cdot 2 + 4 = \boxed{14}$. For a construction that achieves this lower bound, we can take $(a, b, c) = (4, 3, 4)$. □

19. **[6]** Find the number of ordered tuples $(a_1, a_2, \ldots, a_{10})$ that satisfy the following conditions:

- $a_i \in \{1, 2, 3\} \quad (i = 1, 2, \ldots, 10)$.
- If $i = 1, 3, 5, 7, 9$, then $a_i < a_{i+1}$.
- If $i = 2, 4, 6, 8$, then $a_i \geq a_{i+1}$.

Proposed by: Ryan Tang and David Kim

Solution. $\boxed{243}$

Consider grouping the 10 elements into 5 pairs of (a_{2k-1}, a_{2k}) for $k = 1, 2, \ldots, 5$. From the conditions, it follows that only the pairs $(1, 3), (2, 3), (1, 2)$ are possible. Any sequence that is formed by concatenating any 5 of them is valid, so 3^5 is the answer. □

20. **[6]** Let α be a real number picked uniformly at random from $(0,1)$. Find the expected value of

$$\sum_{k=1}^{10} \lfloor k\alpha \rfloor = \lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor 10\alpha \rfloor.$$

Proposed by: Ryan Tang

Solution. $\boxed{\frac{45}{2}}$

Observe that by Linearity of Expectation:

$$\begin{aligned}\mathbb{E}\left[\sum_{k=1}^{10} \lfloor k\alpha \rfloor\right] &= \sum_{k=1}^{10} \mathbb{E}[\lfloor k\alpha \rfloor] \\ &= \sum_{k=1}^{10} \frac{1}{k}(0+1+\cdots+k-1) \\ &= \sum_{k=1}^{10} \frac{1}{k}\frac{k(k-1)}{2} \\ &= \sum_{k=1}^{10} \frac{k-1}{2} \\ &= \boxed{\frac{45}{2}}.\end{aligned}$$

□

21. **[6]** A sphere with center O and radius 169 contains points A, B, and C on its surface such that $AB = 104$, $AC = 112$, and $BC = 120$. Let P_1, P_2, and P_3 be the planes tangent to the sphere at A, B, and C respectively. These three planes intersect at a unique point X. Find the value of OX.

Proposed by: Raymond Xu

Solution. $\boxed{\frac{2197}{12}}$

Because X lies on the tangents, it must be equidistant from A, B, C and thus lies on OO' where O' is the circumcenter of $\triangle ABC$. We also note that X depends only on the distance from $\triangle ABC$ to O, because all tangents intersecting (ABC) must also intersect X.

Then we compute the distance from $\triangle ABC$ to O and get

$$\sqrt{169^2 - 65^2} = 156$$

and from here use similar triangles to get the answer is $\frac{OX}{OA} = \frac{OA}{156} \implies OX = \boxed{\frac{13^3}{12}}$. □

22. **[6]** For each color in the set {red, orange, yellow, green, blue, purple}, there are three balls of that color. Find the number of ways to partition the balls into groups of three such that no group contains balls of three different colors.

Proposed by: Evin Liang, Ryan Tang

Solution. $\boxed{720}$

Note that the possible types are AAA or AAB where A and B are distinct colors. Note that if any group is AAA, then all of those colors are used. Thus, if we pick k numbers to go to itself, then there are $\binom{6}{k}$ ways to pick this. From here, we can't have the colors going to themselves, so we are looking for the number of derangements of $6-k$ (recall that derangements is the number of ways to permute the elements so there are no fixed points). However, using the fact that

$$\sum_{k=0}^{n} \binom{6}{n} \text{\# of derangements} = n!,$$

we get $6! = \boxed{720}$. (The above formula comes from picking which elements are fixed and then permuting the rest). □

23. **[6]** Distinct primes p, q, r satisfy

$$2p^{q-1} + 2q^{p-1} = 3pr + 7qr + 2.$$

Find the value of r.

Proposed by: Samuel Tsui

Solution. $\boxed{37}$

Taking mod p and using Fermat's Little Theorem gives $2 \equiv 7qr + 2 \pmod{p}$, so $p = 7$. Similarly $2 \equiv 3pr + 2 \pmod{q}$, so $q = 3$. Plugging these in gives $r = \boxed{37}$. □

24. **[6]** Let $ABCD$ be a convex quadrilateral with $AB = 9, BC = 15$, and $CD = 13$. Let the diagonals intersect at E. Suppose (AED) and (BCE) intersect AB at points $P \neq A$ and $Q \neq B$. Given $DP \parallel CQ$, find DA.

Proposed by: Ryan Tang

Solution. $\boxed{5}$

The main idea is that the diagonals are perpendicular. To see why, we have that $\angle BQC = \angle AED = 180 - \angle APD$, but $\angle BQC = \angle APD$. Thus, $\angle APD = 90$, so the diagonals are perpendicular. To finish, note that $AB^2 + CD^2 = BC^2 + AD^2$ so $\sqrt{9^2 + 13^2 - 225} = \boxed{5}$ is our answer. □

25. **[6]** Find the value of

$$\sum_{n=0}^{\infty} \frac{2^n}{2026^{2^n} + 1}.$$

Proposed by: Ryan Tang

Solution. $\boxed{\frac{1}{2025}}$

Replace 2026 with X.

$$\frac{2^n}{X^{2^n} + 1} = 2^n \left(\frac{1}{X^{2^n} - 1} - \frac{2}{X^{2^{n+1}} - 1} \right) = \frac{2^n}{X^{2^n} - 1} - \frac{2^{n+1}}{X^{2^{n+1}} - 1}$$

which telescopes. Thus, the answer is $\frac{1}{2025}$. □