LMT Spring 2026 Guts Round Solutions- Set 1

Team Name: ____________________

__________ 1. **[6]** Find $2^0+2^6-2-0-2+6$.

Proposed by: James Wu

Solution. $\boxed{67}$ □

__________ 2. **[6]** Let (a,b,c,d) be a permutation of $(2,0,2,6)$. Given that the distance between (a,b) and (c,d) is an integer, find the distance.

Proposed by: James Wu

Solution. $\boxed{6}$ □

__________ 3. **[6]** There are lily pads in a row numbered 1 to 67, with a frog sitting on pad 1. The frog can jump 6 pads forward or 7 pads backward if a pad exists at the target destination. Find the minimum number of jumps needed for the frog to reach pad 4.

Proposed by: James Wu

Solution. $\boxed{7}$

First notice that if the frog stays on the pads the order it jumps in doesn't really matter. If it jumps x times forward and y times backward, we need

$$6x = 7y + 3$$

The smallest non-negative pair of (x, y) that satisfies this equation is $(4,3)$. This gives a minimum of 7 jumps, which can be achieved by doing all forward jumps first then the backward ones. □

LMT Spring 2026 Guts Round Solutions- Set 2

Team Name: ____________________

__________ 4. **[8]** For each positive integer n, let $f(n)$ be the sum of the n consecutive numbers beginning with n. For example, $f(2)=2+3=5$. Find $f(2026)-f(2025)$.

Proposed by: Ryan Tang

Solution. $\boxed{6076}$

$f(2026) = 2026+\cdots+4051$ while $f(2025) = 2025+\cdots+4049$ so $f(2026)-f(2025) = 4051+4050-2025 = \boxed{6076}$. □

__________ 5. **[8]** The number 509 satisfies the property that its representation in base-4 reads the same forwards and backwards. Find the smallest number greater than 509 which also satisfies this property.

Proposed by: James Wu

Solution. $\boxed{514}$

Note that the base-4 representation of 509 is 13331_4. Since the maximum digit in base 4 is 3, the middle digits can't be incremented, so the next special number has to be at least 20000_4. Since the last digit should match the first, the next special number is 20002_4, which is $\boxed{514}$. □

6. **[8]** Inside a drawer in a dark room, there are 2023 red beads, 2024 white beads, 2025 blue beads, and 2026 yellow beads. Find the minimum number of beads that must be drawn to guarantee drawing at least 1014 beads of a single color.

Proposed by: Larry Cui

Solution. $\boxed{4053}$

If one draws 4052 beads, then it is possible to draw 1013 beads of each color. Thus, by the pigeonhole principle, $\boxed{4053}$ beads are needed in order to force least one color for which 1014 beads are drawn. □

LMT Spring 2026 Guts Round Solutions- Set 3

Team Name:

7. **[10]** A random factor of 10206 is chosen. Find the probability that it is less than 101.

Proposed by: Ryan Tang

Solution. $\boxed{\frac{1}{2}}$

Suppose there exists $x, y \geq 101$ such that $xy = N$. This is obviously impossible. Thus, exactly one of x and $\frac{N}{x}$ are less than 101. This gives the answer of $\frac{1}{2}$. □

8. **[10]** In the addition shown below, A, B, C, D, E, F, G, and H are distinct nonzero digits.

$$\begin{array}{cccccc} & A & B & C & D \\ + & E & F & G & H \\ \hline 1 & 1 & 1 & 1 & 1 \end{array}$$

Find the unique nonzero digit that is not used.

Proposed by: Rohan Danda

Solution. $\boxed{4}$

This problem is the same as checking the set of numbers from 1 to 9 and trying to create 3 pairs that sum to 10 and 1 pair that sums to 11. In total, all numbers sum to 45. So, the remaining digit must be $45 - 11 - 30 = 45 - 41 = \boxed{4}$. □

9. **[10]** Nonzero real numbers x and y satisfy the following system of equations:

$$x + y = 20,$$
$$\frac{1}{x} + \frac{1}{y} = \frac{1}{26}.$$

Find the value of $(x+1)(y+1)$.

Proposed by: Ryan Tang

Solution. $\boxed{541}$

Dividing the second equation from the first gives $xy = 520$. Then,

$$xy + x + y + 1 = 520 + 20 + 1 = \boxed{541}.$$

□

______ 10. **[12]** In isosceles trapezoid $ABCD$ with $AB = CD = 1$, $BC = 3$, and $AD \parallel BC$, let M be the midpoint of diagonal BD. If $\angle ABC = 120°$ and lines AM and BC intersect at E, find the area of quadrilateral $ABED$.

Proposed by: Ryan Tang

Solution. $\boxed{2\sqrt{3}}$

Note that $ABED$ forms a parallelogram because $BE \parallel AD$ and AE bisects BD. Then, note that AD must be $3+2\cdot\frac{1}{2} = 4$ while the altitude from A to BE is $\frac{\sqrt{3}}{2}$. Thus, the answer is $\boxed{2\sqrt{3}}$. □

______ 11. **[12]** A restaurant has ten dishes on its menu. Ben will select three dishes to eat, and he is allowed to select the same dish more than once. Find the number of distinct groups of dishes can Ben choose to eat. Two groups of dishes are considered distinct if and only if some dish is ordered in different quantities between the two groups.

Proposed by: Jerry Xu

Solution. $\boxed{220}$

This is

$$\binom{10}{1} + 2\cdot\binom{10}{2} + \binom{10}{3} = \boxed{220}.$$

□

______ 12. **[12]** Find the value of

$$N = 10497^2 \cdot 2 - 14845^2$$

given that N is between -9 and 9, inclusive.

Proposed by: Ryan Tang

Solution. $\boxed{-7}$

Taking the expression modulo 4, the expression evaluates to 1 while modulo 5, the expression evaluates to $9\cdot 2 - 0 = -2$. Thus, by chinese remainder theorem, it becomes $-7 \pmod{20}$. Thus, the answer is -7. □

LMT Spring 2026 Guts Round Solutions- Set 5

Team Name:

______ 13. **[14]** A circle with center O has radius 3. Segment AB, of length 8, is tangent to the circle at point A. Let M be the midpoint of AB. Let P be the intersection of segment MO and the circle. Find the area of $\triangle BMP$.

Proposed by: Ryan Tang

Solution. $\boxed{\frac{12}{5}}$

Using the Pythagorean theorem, note $MP = 5-3 = 2$. There are a few ways to finish. By the sine area formula, we get the answer of $\frac{3}{5}\cdot 2\cdot 4\cdot \frac{1}{2} = \frac{12}{5}$. Alternatively, if Q is the foot from B to MO, then $BMQ \sim OMA$ (in fact $BQAO$ is cyclic) so $BQ = \frac{12}{5}$ and we are done. □

______ 14. **[14]** Let $N = 121212$. The number n is formed by replacing 3 digits of N with a 9. Find the maximum possible value of $\gcd(n, N)$.

Proposed by: Ryan Tang

Solution. $\boxed{40404}$

The answer is $40404 = \frac{121212}{3}$ achieved by $n = 929292$. Note that this implies that any better solution would have $\gcd(n, N) \in \{N/2, N\}$. However, since $10101 \mid \gcd(n, N)$ in both cases, we can easily see that this is impossible. □

______ 15. **[14]** Find the value of

$$\frac{377^2+89^2}{144^2+233^2}.$$

Proposed by: Ryan Tang

Solution. $\boxed{2}$

Solution. □

LMT Spring 2026 Guts Round Solutions- Set 6

Team Name:

______ 16. **[16]** Two distinct four digit team IDs (without leading zeros) are *similar* if two adjacent digits in one number can be swapped to become the other. For example, 1234 is similar to 1324. LMT doesn't want any pair of team IDs to be similar. Estimate the maximum number of teams LMT can have. If the true answer is A and your answer is E, you will earn $\left\lfloor 16\min\left(\frac{A}{E}, \frac{E}{A}\right)^{10}\right\rfloor$ points.

Proposed by: James Wu

Solution. $\boxed{4590}$

First of all, notice that the multiset of the digits never changes.

After playing around with swapping adjacent digits, one might realize that it's impossible to reach the original number itself in an odd number of adjacent swaps (you can consider the number of inversions to prove this formally). One can therefore color each number black or white, where every number obtainable from a black number in a swap is white and vice versa. Since we want the larger count of black and white numbers when they aren't equal, the number of teams LMT can have is at least

$$\frac{9000}{2} = 4500$$

Better estimates can be obtained by considering cases such as 1120 where the number of IDs that can be obtained from shuffling the digits without having leading zeros is odd. □

__________ 17. **[16]** Your team will pick an integer from 0 to 16 inclusive, and if you picked x, you earn $\max(x-3t,0)$ points, where t is the number of teams other than your team that also picked x. Your answer should be an integer between 0 and 16, inclusive.

Proposed by: Leroy, Isabella

Solution. $\boxed{ans}$

sol □

__________ 18. **[16]** A prime minister oversees 100 households in a line, each having 1 or 2 people with $\frac{1}{2}$ probability. Let M be the maximum number of consecutive regions the households can be divided into so that the population of each region is prime. Estimate the expected value of M. If the true answer is A and your answer is E, you will earn $\max(0,\lceil 16-2|E-A|\rceil)$ points.

Proposed by: James Wu

Solution. $\boxed{\dfrac{599}{9}}$

It's not hard to show that it's always possible to divide the households into contiguous regions with a prime population (just group into twos)

Claim: Given the number of people in each household, let the number of twos be a and the length of the i^{th} contiguous chain of ones be L_i. The answer is

$$a+\sum\left\lfloor\frac{L_i}{2}\right\rfloor.$$

Proof: It is achievable since we can group each pair of adjacent ones into a 2, until we have something like

$$1,2,\ldots,2,1,2,\ldots,2,1,\ldots$$

Where the number of twos exceed the number of ones because 100 is even. Then, it is possible to group each remaining one to a two to form a three, which is prime. This is optimal because it maximizes the number of twos and the remaining group sizes are all threes.

So we can try to find the expected value using linearity of expectation. The expected value of a is $\frac{100}{2}$. By using states, the expected value of the latter is

$$\frac{100}{6}-\frac{1}{9}+\epsilon$$

where $\epsilon = C\cdot 2^{-100}$ for some small C. Therefore, the total expected value is

$$E=50+\frac{100}{6}-\frac{1}{9}+\epsilon=\frac{599}{9}+\epsilon.$$

This gives 665. □

LMT Spring 2026 Guts Round Solutions- Set 7

Team Name:

__________ 19. **[18]** Let a,b,c be numbers satisfying

$$a^6+b^{2027}c=67,$$
$$b^{2028}+ac=7,$$
$$c^2+a^3b=17.$$

Find the sum of all possible values of abc.

Proposed by: Ryan Tang

Solution. $\boxed{0}$

Observe that if (a,b,c) is a solution, then so is $(-a,-b,-c)$. Thus, abc cancels out with $(-a)(-b)(-c) = -abc$, and the answer is 0. □

__________ 20. **[18]** In a regular 200-gon with vertices labelled $P_1P_2\ldots P_{200}$ with consecutive vertices, the lines $P_{11}P_{111}, P_7P_{77}, P_mP_n$ are distinct but concurrent lines. Find the ordered pair (m,n), given that $m > n$.

Proposed by: Ryan Tang

Solution. $\boxed{(145,15)}$

Note that $P_{11}P_{111}$ is a diameter. Then, letting P_7P_{77} and P_mP_n be reflections across the diameter would work. This gives $m = 15, n = 145$, so our answer is $\boxed{(145,15)}$. □

__________ 21. **[18]** Suppose a,b are integers such that $ab = 420$. Find the sum of all possible values of $\frac{a}{b}$.

Proposed by: Ryan Tang

Solution. $\boxed{650}$

Suppose a,b are positive. If they are both negative, they give the same answer. Note that $420 = 2^2\cdot 3\cdot 5\cdot 7$.

Write $b = \frac{140}{a}$, so $\frac{a}{b} = \frac{a^2}{420}$. Hence, the answer is

$$\sum_{a|420}\frac{a^2}{420} = \frac{(1+2^2+4^2)(1^2+3^2)(1+5^2)(1+7^2)}{420} = \frac{21\cdot 10\cdot 26\cdot 50}{420} = \boxed{650}.$$

□

. . . .

______ 22. **[20]** For every positive integer k, define

$$a_k = \sqrt{1 + \frac{1}{k^2} + \frac{1}{(k+1)^2}}.$$

Find the value of the sum

$$a_1 + a_2 + \cdots + a_{99} + a_{100}.$$

Proposed by: Peter Bai

Solution. $\boxed{\frac{10200}{101}}$

We will begin by manipulating the definition of a_k in the hopes of finding a more usable form. After some expanding, we get

$$a_k = \sqrt{\frac{k^2(k+1)^2 + (k+1)^2 + k^2}{k^2(k+1)^2}} = \sqrt{\frac{k^4 + 2k^3 + 3k^2 + 2k + 1}{k^2(k+1)^2}} = \sqrt{\frac{(k^2+k+1)^2}{k^2(k+1)^2}} = \frac{k^2+k+1}{k(k+1)}.$$

This is much nicer! Next, we want to add together all of the a_k, so we might begin looking for some sort of telescoping idea. Indeed, a_k seems to be very close to 1, which motivates rewriting it to be 1 plus something:

$$a_k = \frac{k^2+k+1}{k^2+k} = 1 + \frac{1}{k^2+k} = 1 + \frac{1}{k(k+1)} = 1 + \frac{1}{k} - \frac{1}{k+1}.$$

The last two terms will telescope when summed over consecutive integers, so we can now proceed with evaluating the sum:

$$\begin{aligned}
&\left(1 + \frac{1}{1} - \frac{1}{2}\right) + \left(1 + \frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(1 + \frac{1}{100} - \frac{1}{101}\right) \\
=&100 \cdot 1 + \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{100} - \frac{1}{101} \\
=&100 + 1 - \frac{1}{101} \\
=&\frac{101^2 - 1}{101} \\
=&\boxed{\frac{10200}{101}}.
\end{aligned}$$

□

______ 23. **[20]** Find the number of 7-letter words formed by the letters of JEOPARDY (repetition allowed) such that the word JERRY can be obtained by deleting exactly two letters from the words.

Proposed by: James Wu

Solution. $\boxed{1079}$

For each slot, we have 1 way of getting the target letter we want and 7 ways to get a different letter. We want the number of successes to be at least 5, so the answer is

$$\binom{7}{5} \cdot 7^2 + \binom{7}{6} \cdot 7^1 + \binom{7}{7} \cdot 7^0 = 1029 + 49 + 1 = \boxed{1079}.$$

□

______ 24. **[20]** The expression

$$\frac{1}{n^4}\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j}\sum_{\ell=1}^{k} 1$$

approaches a real number L as n approaches infinity. Find L.

Proposed by: Atticus Oliver

Solution. $\boxed{\frac{1}{24}}$

This is effectively counting $\frac{1}{n^4}$ times the number of ways to take an ordered quadruple (i, j, k, ℓ) such that $n \ge i \ge j \ge k \ge \ell$. Notice that the coordinates i, j, k, ℓ can be arbitrarily rearranged to get a different ordered quadruple which will not satisfy the condition if $i \ne j \ne k \ne \ell$. As n approaches infinity, the number of cases where any pair of the coordinates are equal grows insignificant, implying that $\frac{1}{24}$ of ordered quadruples with each coordinate an integer between 1 and n inclusive will work for large n, giving $\frac{n^4}{24}$ cases. The leading division by n^4 gives the desired $\boxed{\frac{1}{24}}$. □

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LMT Spring 2026 Guts Round Solutions- Set 9

Team Name: ______

______ 25. **[22]** Find the value of

$$\sqrt{4^{2026}+\sqrt{4^{2027}+\sqrt{4^{2028}+\dots}}}.$$

Proposed by: Raymond Xu

Solution. $\boxed{2^{2026}+1}$

$$x+1=\sqrt{x^2+2x+1}=\sqrt{x^2+\sqrt{4x^2+4x+1}}=\sqrt{x^2+\sqrt{4x^2+\sqrt{16x^2+8x+1}}}=\dots$$ □

______ 26. **[22]** Let $ABCD$ be a convex quadrilateral with $AB = CD$ and $AD < BC$. Suppose AB and CD intersect at X and the circles (XAC) and (XBD) intersect along line BC. Given that $BX = 6, CX = 7, BC = 8$, find $AB + CD$.

Proposed by: Ryan Tang

Solution. $\boxed{\frac{128}{13}}$

The main idea is that the concurrency point is the foot of the angle bisector of $\angle BXC$. Let Y be the concurrency point. Indeed, by POP at B and POP at C

$$BA \cdot BX = BY \cdot BC$$
$$CD \cdot CX = CY \cdot CB.$$

Dividing the two, we get $\frac{XB}{BY} = \frac{XC}{CY}$ which is exactly the statement of angle bisector theorem. We get that $BY = \frac{6}{13} \cdot 8$, so $AB = \frac{BC \cdot BY}{XB} = \frac{8 \cdot 6 \cdot 8}{13 \cdot 6} = \frac{64}{13}$. This gives $\boxed{\frac{128}{13}}$. □

______ 27. **[22]** An interval is formed by selecting two endpoints chosen independently and uniformly at random in the interval $(0,1)$. Two real numbers are then selected in the interval $(0,1)$, also independently and uniformly at random. Given that both of these numbers lie within the chosen interval, find the probability that $\frac{2}{3}$ also lies within the chosen interval.

Proposed by: Edwin Zhao

Solution. $\boxed{\frac{64}{81}}$

note this is essentially the same as choosing 4 points where the interval is just the smallest and the largest point so given four random a, b, c, d compute probability $\max(a, b, c, d) > \frac{2}{3}$ and $\min(a, b, c, d) < \frac{2}{3}$ so it's js $1 - \left(\frac{1}{3}\right)^4 - \left(\frac{2}{3}\right)^4 = \boxed{\frac{64}{81}}$ □

LMT Spring 2026 Guts Round Solutions- Set 10

Team Name: ______________________

The following round is a cyclic round. In the following three problems, let a be the answer to the first problem, b the answer to the second problem, and c the answer to the third problem. It is given that all of these answers are positive integers.

__________ 28. **[23]** Find the sum of all possible real $x + y$ that satisfy the following system of equations:

$$\begin{cases} by + cx = -5b \\ (x-b)^2 + (y+c+5)^2 = (bc)^2 \end{cases}$$

Proposed by: James Wu

Solution. $\boxed{4}$

The first equation is a line and the second one is a circle, so we want to find the sum of the coordinates of the intersections. Notice that the center of the circle in the second equation, $(b, -c-5)$, satisfies the first equation. This means that the line passes through the center, so the sum of the coordinates of the intersections is $2(b - c - 5) = 2b - 2c - 10$. □

__________ 29. **[23]** Suppose that there is a grid with dimensions $a \times c$, where each cell is black or white. A grid is good if it can be turned into all black by repeatedly toggling 2×2 subgrids. Let N be the number of good grids. Find the sum of the exponents in the prime factorization of N.

Proposed by: James Wu

Solution. $\boxed{12}$

Assuming that $a, c \geq 2$, the grid would have $(a-1)(c-1)$ different 2×2 subgrids. Therefore, there will be at most $2^{(a-1)(c-1)}$ good grids. In fact, there is exactly $2^{(a-1)(c-1)}$ good grids. It is sufficient to prove that if we have a completely black grid, it is impossible to keep everything black after toggling a non-empty set of subgrids.

If we consider the top-left corner, it can only be toggled by the top-left subgrid, so the top-left subgrid must not be toggled. After this, we can consider the cell on the first row and second column, then the only possibly toggled subgrid that touches the cell must not be toggled as well. In a similar manner, we can prove that all subgrids must not be toggled if we want to keep the grid entirely black. Thus, our answer is $(a-1)(c-1)$. □

__________ 30. **[23]** Let $ADMITS$ be an equiangular hexagon centered at L such that $\overline{AD} = \overline{IT} = b$ and $\overline{DM} = \overline{MI} = \overline{TS} = \overline{SA} = a$. Given that the area of triangle $\triangle LMT$ can be expressed as $x\sqrt{y}$, where y is squarefree, find $x - y$.

Proposed by: James Wu

Solution. $\boxed{5}$

The quadrilateral $MITS$ can be cut into an equilateral triangle and a parallelogram. Since $MITS$ is a trapezoid, the parallelogram and the triangle have the same height of $\frac{a\sqrt{3}}{2}$. The equilateral triangle has base length a and parallelogram base length b, summing to $a+b$. Thus, the area of triangle $\triangle LMT$ is

$$\frac{1}{2}\cdot\frac{a+b}{2}\cdot\frac{a\sqrt{3}}{2}=\frac{a(a+b)}{8}\sqrt{3}$$

Therefore, $x-y=\frac{a(a+b)}{8}-3$. □

We can try to combine the equations from problem 1 and 2 first. Consider

$$\begin{cases} a=2b-2c-10 \\ b=(a-1)(c-1) \end{cases}$$

We can substitute b in the second equation with $\frac{a+2c+10}{2}$ and multiply the entire equation by 2. Then, after expanding we would have

$$2ac-3a-4c=8$$

Or

$$(a-2)(2c-3)=14$$

From problem 2, we can assume that a,b,c are all positive integers. Therefore, the only possible solutions are $(a,b,c)=(4,12,5)$ and $(16,15,2)$. However, the second one doesn't satisfy the equation for problem 3, so the answer is $(a,b,c)=\boxed{(4,12,5)}$

. . . .

______ 31. **[24]** Let P_k denote the probability that Ben will not get any duplicate values when he rolls k fair, 120-sided dice each with faces labeled using the integers from 1 to 120, inclusive. There exists an integer n such that $P_n - P_{n+1}$ is maximized. Find n.

Proposed by: Peter Bai

Solution. $\boxed{11}$

We begin by finding an expression for the value of P_k. Trivially, $P_1 = 1$ because it is impossible to get a duplicate value with one die. If we roll another die, then the probability that we do not get a duplicate value is 119/120, because any value other than the one from the first die works. By analogous logic, the third die does not get us duplicate values with probability 118/120, and we can continue in this manner to show that

$$P_k = \frac{120}{120}\cdot\frac{119}{120}\cdots\frac{120-k+1}{120}.$$

Our next insight will be to notice that $P_n - P_{n+1} > P_{n-1} - P_n$ if and only if the following inequality holds:

$$\frac{P_n - P_{n+1}}{P_{n-1} - P_n} > 1.$$

Using this fact, we now just need to find the largest n where this inequality holds—this would be the last value of n for which $P_n - P_{n+1}$ increases, after which it will begin to decrease. (We are tacitly assuming that this ratio will never become 1 again after it falls below 1, which we can later see to be true after we do some algebra.)

By plugging in the expression for P_n obtained earlier, we can expand our inequality as follows:

$$\frac{\frac{120}{120}\cdot\frac{119}{120}\cdots\frac{120-n+1}{120} - \frac{120}{120}\cdot\frac{119}{120}\cdots\frac{120-n}{120}}{\frac{120}{120}\cdot\frac{119}{120}\cdots\frac{120-n+2}{120} - \frac{120}{120}\cdot\frac{119}{120}\cdots\frac{120-n+1}{120}} > 1.$$

Thankfully, there's a lot we can do to simplify the left hand side:

$$\begin{aligned}
\frac{\frac{120}{120}\cdot\frac{119}{120}\cdots\frac{120-n+1}{120}\left(1-\frac{120-n}{120}\right)}{\frac{120}{120}\cdot\frac{119}{120}\cdots\frac{120-n+2}{120}\left(1-\frac{120-n+1}{120}\right)} &> 1\\
\frac{\frac{120-n+1}{120}\left(1-\frac{120-n}{120}\right)}{1-\frac{120-n+1}{120}} &> 1\\
\frac{(121-n)(n)}{120n-120} &> 1\\
\frac{-n^2+121n}{120n-120} &> 1\\
\frac{-n^2+n+120}{120n-120} &> 0\\
-n^2+n+120 &> 0.
\end{aligned}$$

The largest integer value of n for which this is true is $n = \boxed{11}$. □

______ 32. **[24]** Let ABC be a triangle with $AB = 1$, $BC = \sqrt{2}$, and $\angle ABC = 90^\circ$. Let point D be on the same side of line AC as B such that $\angle DAC = 90^\circ$ and $AD = AB$. Finally, let line AB intersect the circumcircle of $\triangle ADC$ at a point E different from A. Find the length of BE.

Proposed by: Peter Bai

Solution. $\boxed{\frac{\sqrt{6}}{3}}$

Denote the circumcircle of $\triangle ADC$ as Ω. Then, extend side BC to meet Ω at a new point P different from C. By inscribed angles, we have

$$\angle PAB = \angle DAB - \angle DAP = (90^\circ - \angle BAC) - \angle DCP = \angle ACB - \angle DCP = \angle ACD,$$

so $\triangle PBA \sim \triangle DAC$. By Pythagoras, $AC = \sqrt{(1)^2 + (\sqrt{2})^2} = \sqrt{3}$, so

$$\frac{PB}{AB} = \frac{DA}{CA} \implies \frac{PB}{1} = \frac{1}{\sqrt{3}} \implies PB = \frac{\sqrt{3}}{3}.$$

Finally, by Power of a Point about B, we get

$$PB \cdot BC = AB \cdot BE \implies \frac{\sqrt{3}}{3} \cdot \sqrt{2} = 1 \cdot BE \implies BE = \boxed{\frac{\sqrt{6}}{3}}.$$

□

______________ 33. **[24]** Find all real numbers k such that

$$20x^4 - 26x^3 + kx^2 - 26x + 20 = 0$$

has exactly three distinct real solutions in x.

Proposed by: James Wu

Solution. $\boxed{-92}$

Let $f(x) = 20x^4 - 26x^3 + kx - 26x + 20$. Notice that

$$f(x) = x^4 f\left(\frac{1}{x}\right)$$

So for every solution $r \neq 0$, $\frac{1}{r}$ is also a solution. Since the number of solutions is odd, either 1 or -1 must be a solution.

If $x = 1$ is a solution, we have $k = -(20 - 26 - 26 + 20) = 12$. If $x = -1$ is a solution, we have $k = -(20 + 26 + 26 + 20) = -92$.

However, notice that by Vieta's formulas, the sum of roots is $\frac{26}{20} = \frac{13}{10}$. And when 1 is a solution, we must have

$$2 + r + \frac{1}{r} = \frac{13}{10}$$

for some real r satisfying $f(r) = 0$. Since $\left|r + \frac{1}{r}\right| \geq 2$, $k = 12$ doesn't work. Thus, the only real number k satisfying the condition is $k = \boxed{-92}$. □

LMT Spring 2026 Guts Round Solutions- Set 12

Team Name: ______________

______________ 34. **[27]** For a nonnegative integer n, let $f(n)$ denote the number of ordered pairs of nonnegative integers (a, b) satisfying

$$a^2 + b^2 = n.$$

Estimate the sum of the 10001 terms in the sequence

$$\left\{f\left(2026^2 - 5000\right), f\left(2026^2 - 4999\right), f\left(2026^2 - 4998\right), \ldots, f\left(2026^2 + 5000\right)\right\}.$$

If the true answer is A and your answer is E, you will earn $\left\lfloor 27\left(\frac{A}{E}, \frac{E}{A}\right)^{67} \right\rfloor$ points.

Proposed by: Peter Bai

Solution. $\boxed{7842}$

Our first insight will be to notice that $a^2+b^2=N$ is the equation for a circle centered at the origin with radius $\sqrt{N}$. Consequently, our answer will be the number of lattice points (a,b) with $a,b\ge 0$ that lie between the circles with $r_1=\sqrt{2026^2-5000}$ and $r_2=\sqrt{2026^2+5000}$. Approximating this as the difference in area between the quarter circles with these two radii, we get

$$\frac{\pi}{4}r_2^2-\frac{\pi}{4}r_1^2=\frac{\pi}{4}(r_2^2-r_1^2)=\frac{\pi}{4}\cdot 10001\approx\frac{31416}{4}=\boxed{7854},$$

which is enough for 24 points. □

35. **[27]** You have a 4×4 grid in which you can fill in numbers 1 to 8, using each number exactly twice. A downright path is considered *valid* if the sum of the numbers on the path is prime. The score of a valid path is the sum of the numbers on the path. Estimate the maximum sum of scores over all valid paths on your grid. If the true answer is A and your answer is E, you will earn $\left\lfloor 27\min\left(\left(\frac{E}{A}\right)^{14},\left(\frac{A}{E}\right)^{34}\right)\right\rfloor$ points

Proposed by: James Wu

Solution. $\boxed{721}$

Define the frequency $f(x,y)$ as the number of valid paths passing through cell (x,y). We can generate the following frequency map:

20	10	4	1
10	12	9	4
4	9	12	10
1	4	10	20

We want the larger numbers to have higher frequencies, so we can create a grid like

8	5	2	1
6	7	4	2
3	4	7	5
1	3	6	8

This gives a score of **113**. We can optimize this to **201** by swapping a few numbers around, but this still isn't that good.

To obtain a better construction, we have to look at parity. If we fill in the grid with a checkerboard pattern, every path traverses 4 cells of one color and 3 cells of another. To guarantee that every path sum is odd, we need to place all 8 even numbers on the 4-cell positions and all 8 odd numbers on the 3-cell positions. If we combine this idea and the frequency map, we can create a grid like

8	5	4	1
7	6	3	4
2	3	6	5
1	2	7	8

which gives a score of **492**. With the following more optimal construction by noticing that each path goes through each anti-diagonal once

8	7	4	1
7	6	3	2
4	3	6	5
1	2	5	8

we can obtain a score of **582**, which is already pretty good. The anti-diagonal observation leads to our final optimization: since our previous step guaranteed that no paths are a multiple of 2, we can also try to construct the grid to guarantee that no paths are a multiple of 3. Specifically, we can come up with the following construction

8	3	2	1
3	8	7	4
2	7	6	5
1	4	5	6

which gives a score of **714**, only 7 away from the optimal score of **721** given by the construction

6	5	2	1
5	8	3	2
4	7	8	7
3	4	1	6

□

______ 36. **[27]** Estimate the number of teams that correctly answered Guts 19, of which the answer was 0. If the true answer is A and your answer is E, you will earn $\max(0, \lfloor 27 - 9|E - A| \rfloor)$ points. If you have forgotten, Guts 19 is provided below:

> Let a, b, c be numbers satisfying
>
> $$a^6 + b^{2027}c = 67,$$
> $$b^{2028} + ac = 7,$$
> $$c^2 + a^3 b = 17.$$
>
> Find the sum of all possible values of abc.

Proposed by: Ryan Tang

Solution. $\boxed{ans}$

sol □

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