Team Round Solutions

LMT Spring 2025

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1. **[20]** Find the sum of all integers x for which $|x-5|+|x-10|$ is minimized.

   Proposed by: Peter Bai, Charles Chen, Chris Cheng, Rohan Danda, Alexander Duncan, Henry Eriksson, Adam Ge, Selena Ge, William Hua, Isabella Li, Jonathan Liu, Atticus Oliver, Alfonso Quinones, Muztaba Syed, Ryan Tang, Samuel Tsui, Micah Wang, Samuel Wang, James Wu, Danyang Xu, Jacob Xu, Jerry Xu, Edwin Zhao

   Solution. $\boxed{45}$

   This is the sum of the distances from x to 5 and 10. Thus x should be between them to get a minimum value of 5. Since x is an integer, it can be any of $\{5, 6, 7, 8, 9, 10\}$. The sum of these values is $\boxed{45}$. □

2. **[25]** Suppose x is a positive integer satisfying $100^2+2^2=20^2+x^2$. Find x.

   Proposed by: Jacob Xu

   Solution. $\boxed{98}$

   Note that
   $$(100-2)^2=100^2-2\cdot 2\cdot 100+2^2=100^2-20^2+2^2.$$
   Thus the answer is $\boxed{98}$. □

3. **[25]** Jordan sums the numbers from 1 to 100 inclusive. However, he accidentally excludes two numbers from the sum and gets a multiple of 97. Find the maximum possible sum of these two numbers.

   Proposed by: Benjamin Yin

   Solution. $\boxed{103}$

   $1+2+3+\cdots+100=5050\equiv 6 \pmod{97}$. So, the sum of the two numbers is $6 \pmod{97}$ which gives possible sums of 6 and 103, thus the final answer is $\boxed{103}$. □

4. **[30]** Find the number of permutations a_1, a_2, a_3, a_4, a_5 of the numbers $2,3,4,5,6$ such that $\frac{a_k}{k}$ is an integer for all $1\le k\le 5$.

   Proposed by: Jonathan Liu

   Solution. $\boxed{3}$

   Note that $a_5=5$ and $a_4=4$ because these are the only numbers that guarantee that $\frac{a_5}{5}$ and $\frac{a_4}{4}$ are integers. Next, we can consider which number a_2 corresponds to.

   Because $\frac{a_2}{2}$ is an integer, and since we are left with $2,3,6$, a_2 equals 2 or 6.

   Case 1: $a_2=2$ In this case, $(a_1,a_3)=(6,3)$ and $(a_1,a_3)=(3,6)$ both work.

   Case 2: $a_2=6$ In this case, only $(a_1,a_3)=(2,3)$ works.

   Thus, we have a total of $2+1=\boxed{3}$ cases. □

5. **[30]** Primes p, q, r, s satisfy the following equations

$$3pq - r = 51s,$$
$$5qr - p = 50s, \text{ and}$$
$$7rp - q = 49s.$$

Find (p, q, r, s).

Proposed by: Samuel Tsui

Solution. $\boxed{(5,7,3,2)}$

Note that $3 \mid 3pq - r$, $5 \mid 5qr - p$, $7 \mid 7rp - q$ so $r = 3$, $p = 5$, $q = 7$. Plugging this back in gives $s = 2$ which is a prime. Therefore the answer is $\boxed{(5,7,3,2)}$ □

6. **[35]** William has a bag of red and blue marbles. If he draws a red marble, the probability of drawing a blue marble increases by $\frac{1}{2025}$. Find the least possible number of blue marbles in the bag.

Proposed by: Evin Liang

Solution. $\boxed{52}$

Let the initial number of blue marbles be n and the initial total number of red marbles be m. Then we have $\frac{n}{m-1} = \frac{n}{m} + \frac{1}{2025}$, so $2025n = m(m-1)$. Since $2025 = 5^2 \cdot 3^4$, So one of m and $m-1$ must be divisible by 25 and one must be divisible by 81. Testing gives the least possible value of m is 325, so $n = \frac{324 \cdot 325}{2025} = \boxed{52}$. □

7. **[35]** Let $ABCD$ be a rectangle, and P a point on the circumcircle of $ABCD$. Suppose $\frac{PA}{PD} = \frac{3}{5}$ and $\frac{PB}{PC} = \frac{7}{8}$. Find $\frac{PA}{PC}$.

Proposed by: Muztaba Syed

Solution. $\boxed{\dfrac{3\sqrt{15}}{32}}$

Let $PA = 3x$, $PD = 5x$, $PB = 7y$, $PC = 8y$. Then by the Pythagorean Theorem on APC and BPD we have

$$9x^2 + 64y^2 = 25x^2 + 49y^2 \implies \frac{x}{y} = \frac{\sqrt{15}}{4}.$$

The answer is then $\frac{\sqrt{15}}{4} \cdot \frac{3}{8} = \boxed{\dfrac{3\sqrt{15}}{32}}$. □

8. **[40]** A four digit base 10 number $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}$ (potentially with leading zeroes) is a multiple of 99. Suppose that one of the numbers $\underline{a}\,\underline{b}$ and $\underline{c}\,\underline{d}$ divides the other. Find the sum of all possible values of $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}$.

Proposed by: Muztaba Syed

Solution. $\boxed{69993}$

When we add 99 to a multiple of 99 note that $\underline{a}\,\underline{b}$ increases by 1 and $\underline{c}\,\underline{d}$ decreases by 1. So their sum is 99. The only exception to this is 9999. This tells us that one of the two numbers divides 99, so the numbers can be either a divisor of 99 or 99 minus the divisor.

Thus the sum for each of these is $99 \cdot \tau(99)$. We need to multiply by 101, and then add the 9999 case. In total this is $99 \cdot 101 \cdot (6+1) = (100^2 - 1) \cdot 7 = \boxed{69993}$. □

9. **[45]** Neel travels through all vertices of a regular 21-gon by traveling along diagonals (not sides) and finishes at the vertex where he started. Find the difference between the maximum and minimum number of times his path can intersect itself in the interior of the 21-gon.

Proposed by: Muztaba Syed

Solution. $\boxed{168}$

There are 21 total segments drawn, and adjacent segments can't intersect each other. This gives us an upper bound of at most $\frac{21\cdot 18}{2} = 189$ points of intersection. If the polygon is $A_1A_2\dots A_{21}$, this can be achieved by traveling from A_i to A_{i+10} every move.

For the lower bound a segment cannot be a side length, so it splits the other vertices into two nonempty groups. The path must go from one group to another at least twice, so there are at least 21 intersections. This can be achieved by traveling from A_i to A_{i+2} every move.

Thus our answer is $189-21=\boxed{168}$. □

10. **[45]** Let $ABCD$ be a trapezoid with perpendicular diagonals and $AB \parallel CD$. Let E and F be the feet of the perpendiculars from B to CD and E to BD, respectively. Given that F is the centroid of triangle ADE and $AD = 10$, find the area of the trapezoid.

Proposed by: Adam Ge, Ryan Tang

Solution. $\boxed{75\sqrt{2}}$

Since F is the centroid, DF bisects $\overline{AE}$ so BD passes through the midpoint of $\overline{AE}$. Thus $\triangle ABX \cong \triangle EDX$ where $X = BD \cap AE$, so $ABED$ is a parallelogram. Since $\angle BED = 90$ degrees, $ABED$ is a rectangle. By centroid properties, $DF : FB = 1 : 2 \implies AD : DE = \sqrt{2} : 1$. Thus from $AD = 10$ we get $DE = AB = 5\sqrt{2}$. Since the diagonals are perpendicular, $AB \cdot CD = AD^2 \implies CD = 10\sqrt{2}$ so the area is $\boxed{75\sqrt{2}}$. □

11. **[50]** Real numbers x and y satisfy the following equations:

$$\sqrt{x^2+y^2}+\sqrt{(x-3)^2+(y-4)^2}=5,$$
$$2x^2+3xy=24.$$

Find $(x+y)^2$.

Proposed by: Jonathan Liu

Solution. $\boxed{\dfrac{196}{9}}$

Let $A=(0,0), B=(x,y), C=(x-3,y-4)$ Then $\sqrt{x^2+y^2}+\sqrt{(x-3)^2+(y-4)^2}=5$ tells us that $AB+AC=5$, but by the triangle inequality we know $AB+AC \geq BC = 5$ so therefore C lies on the line from A to B.

Equating slopes, $\frac{y}{x} = \frac{y-4}{x-3}$ so $y = \frac{4}{3}x$

Plugging in to the second equation, we get $6x^2 = 24$ or $x = \pm 2, y = \pm\frac{8}{3}$, so $(x+y)^2 = (\pm\frac{14}{3})^2 = \boxed{\dfrac{196}{9}}$. □

12. **[50]** Mickey Mouse is walking on the Cartesian Plane. He is at the point $(5,0)$ and is trying to get to his biology classroom at $(0,0)$. In each part of his journey, he chooses a lattice point and walks to the lattice point in a straight line. Once he arrives at that lattice point, he chooses another lattice point and repeats the process. Given that Mickey Mouse must be walking closer to his biology classroom at all times, find the maximum distance he travels on his journey.

Proposed by: Alexander Duncan

Solution. $\boxed{\sqrt{5}+3\sqrt{2}+6}$

If A is the lattice point Mickey Mouse is at, B is the lattice point Mickey Mouse is walking to, and O is where Mickey Mouse's biology classroom is, then $\angle ABO \geq 90^\circ$ as otherwise Mickey Mouse will be getting farther from his biology classroom at some point. Therefore, one of the optimal paths is $(5,0) \to (4,2) \to (4,0) \to (3,1) \to (3,0) \to (2,1) \to (2,0) \to (1,1) \to (1,0) \to (0,0)$ which has a total distance of $\boxed{\sqrt{5}+3\sqrt{2}+6}$. □

13. **[55]** Mira partitions the cells of a 5×5 grid into disjoint rectangles. Any vertex on the border (excluding corners) of the 5×5 grid is shared by exactly two rectangles, and any vertex of the 5×5 grid in the interior is shared by exactly 3 rectangles. Find the number of partitions Mira can make.

Proposed by: Muztaba Syed

Solution. $\boxed{36}$

We can think of this partition as removing edges from the 5×5 grid. The condition about interior vertices is just saying that we remove exactly one edge from each vertex on the interior. Vertices on the outside have no edges removed from them. Thus we can just focus on the 4×4 grid of interior vertices.

They are split into pairs where there is no edge between each pair. This is equivalent to splitting a 4×4 grid into 8 2×1 rectangles. We can calculate the number of ways to do this with casework on what happens to the middle 4 vertices.

If they are connected to nothing on the outside, there are $2 \cdot 2$ cases. If two of them are connected to cells around them, there are $4 \cdot 2 \cdot 2$ cases. If all four are connected to vertices on the outside, there are 2^4 options. In total the answer is $4 + 16 + 16 = \boxed{36}$. □

14. **[55]** Let $ABCD$ be a quadrilateral with $\angle B = \angle D = 90°$, and $\angle BCA = 30°, \angle DCA = 45°$. Let M be the midpoint of AC and ω be the circumcircle of BMD. The extensions of lines AB, BC, CD, DA intersect ω again at points W, X, Y, Z, respectively. Find $\frac{AB+AD}{WY+XZ}$.

Proposed by: Henry Eriksson

Solution. $\boxed{\dfrac{\sqrt{2}+1}{2}}$

Define E, F to be the intersections of AB and CD, and AD and CB, respectively (complete $ABCD$). Consider the triangle EFC. $EB \perp FC$, and similarly for FD and EC. This means that EB, FD are altitudes of the triangle, so A is the orthocenter, and B, D are feet of altitudes. ω passes through the midpoint of AC, the point B, and the point D, so ω is the nine-point circle of EFC. This means that X, Y are the midpoints of CE, CF, respectively, and that W, Y are the midpoints of AE, AF, respectively. So WY is the midline of CAE, so $WY = \frac{AC}{2}$. Therefore, $WY + XZ = AC$. $\sin 30 = \frac{AB}{AC}$, and $\sin 45 = \frac{AD}{AC}$, so $\frac{AB+AD}{WY+XZ} = \frac{AB+AD}{AC} = \sin 30 + \sin 45 = \frac{1}{2} + \frac{\sqrt{2}}{2} = \frac{1+\sqrt{2}}{2}$. □

15. **[60]** In math team, every pair of members are either bros or opps. If the previous person is a bro, then a math team member will make a statement of the same truth value, and if the previous person is an opp, then a math team member will make a statement of the opposite truth value. The five captains of math team have the following conversation:

- William: I'm Muztaba's opp.
- Ella: Among me, Muztaba, and Jacob there are an even number of pairs of opps.
- Jacob: Peter and William are opps.
- Muztaba: Jacob has no opps.
- Ella: Jacob has three opps.
- Peter: I'm Jacob's opp.
- William: Peter has no opps.
- Peter: Jacob and Ella are not opps.
- Muztaba: I have more opps than Jacob.

Let W, E, J, M, and P be the number of opps of William, Ella, Jacob, Muztaba, and Peter, respectively. Find $10000W + 1000E + 100J + 10M + P$.

Proposed by: Evin Liang

Solution. $\boxed{32333}$

Note that if Ella's first statement is true, then Ella's second statement must be true, and if Ella's first statement is false, then Ella's second statement must be true. Hence Ella's second statement is true, so Muztaba's first statement is false, so Muztaba and Ella are opps. So from Peter's statement, we get that Peter opps with exactly one of Jacob and Ella, so William's second statement is false. Also, Peter's two statements must have the same truth value so Jacob opps with Peter if and only if he does not opp with Ella. Since Jacob has three opps he must opp with both Muztaba and William. Thus Jacob's statement is true. Thus Peter's statements are both true, and thus Ella and Peter are not opps and Peter opps with Jacob. Now if Muztaba and Peter are not opps, then Muztaba must have four opps, which is impossible. So Peter and Muztaba are opps, and so Muztaba does not opp with William. Thus William's first statement is false, so William and Ella are opps. □