Speed Round Solutions

LMT Spring 2025

May 3, 2025

1. **[6]** If $2025x + 5202 = 2025 + 5202x$, find x.

   Proposed by: Jacob Xu

   Solution. $\boxed{1}$

   By doing the algebra out, we see that $5202 - 2025 = (5202 - 2025)x$, so $\boxed{x = 1}$. □

2. **[6]** A triangle with integer side lengths has perimeter 30, and one side of length 10. Find the maximum possible length of the longest side of this triangle.

   Proposed by: Jacob Xu

   Solution. $\boxed{14}$

   Having side lengths $5, 10, 15$ fails by the triangle inequality because $5 + 10 = 15$, so we increase the shortest side by 1 and decrease the longest side by 1. Thus, the answer is $15 - 1 = \boxed{14}$. □

3. **[6]** Two standard 4-sided dice are rolled. Find the probability their product of the numbers rolled is a square number.

   Proposed by: Jacob Xu

   Solution. $\boxed{\frac{3}{8}}$

   There are two cases. The first is if the two numbers rolled are the same, which gives 4 outcomes. The second is if a 1 and a 4 is rolled, which gives 4 outcomes. So the answer is $\frac{6}{16} = \boxed{\frac{3}{8}}$. □

4. **[6]** A three digit number is formed with digits $1, 2, 3$, each used once, so that the first two digits form a prime number and the last two digits form a prime number. Find this three digit number.

   Proposed by: Jacob Xu

   Solution. $\boxed{231}$

   2 must be first, otherwise there would be an even two digit number formed. Then 3 must be after since 21 is not prime. So the answer is $\boxed{231}$. □

5. **[6]** Topher is eating pickles. He eats a pickle at 12:00 AM midnight on Monday, and from then on, he eats a pickle every 45 minutes until he eats his last pickle on Friday at 4:30 PM. Find the number of times Topher eats a pickle.

   Proposed by: Christopher Cheng

   Solution. $\boxed{151}$

   Each day Topher eats $24 \cdot \frac{60}{45} = 32$ pickles. After 4 days he eats $32 \cdot 4 = 128$ pickles. In the last 16.5 hours, he eats $16.5 \cdot \frac{60}{45} = 22$ pickles. Finally, we need to include the initial pickle he eats at 12:00 AM, for a total of $1 + 128 + 22 = \boxed{151}$. □

6. **[6]** Jon is traveling 60 miles from Lexington High School to Lake Orz. If he travels at a rate of 15 mph for the first 30 miles, and 10 mph for the last 30 miles, find his average speed throughout the entire trip, in mph.

Proposed by: Jonathan Liu

Solution. $\boxed{12}$

It takes him a total of $\frac{30}{15}+\frac{30}{10}=5$ hours for the entire trip. His average speed will be $\frac{60}{5}=\boxed{12}$ miles per hour. □

7. **[6]** Little John's right triangular room has two sides with length 20 and 25. Find the length of the last side if the area of his room is 150 square units.

Proposed by: Atticus Oliver

Solution. $\boxed{15}$

Notice that the 15-20-25 right triangle has area 150, so the last side has length $\boxed{15}$. □

8. **[6]** Equilateral triangle ABC has side length $AB=1$. Point D lies on line BC such that $BD=2025$ and C lies inside segment BD. Find the area of triangle ACD.

Proposed by: Peter Bai

Solution. $\boxed{506\sqrt{3}}$

Notice that triangles $\triangle ABC$ and $\triangle ACD$ share the same altitude from A to line BC (which is the same as line CD). If we let $[\triangle ABC]$ denote the area of triangle $\triangle ABC$, then this means that

$$\frac{[\triangle ABC]}{[\triangle ACD]}=\frac{BC}{CD}$$

which is the ratio of the lengths of their bases. From the problem statement, we can deduce that $BC=1$ and $CD=2025-1=2024$. Additionally, the area of unit equilateral triangle $\triangle ABC$ is $\frac{\sqrt{3}}{4}$. Plugging everything in gives

$$\frac{[\triangle ABC]}{[\triangle ACD]}=\frac{BC}{CD}\implies\frac{\frac{\sqrt{3}}{4}}{[\triangle ACD]}=\frac{1}{2024}\implies[\triangle ACD]=2024\cdot\frac{\sqrt{3}}{4}=\boxed{506\sqrt{3}}.$$

□

9. **[6]** Leo writes the number 1 on the board. Every minute, he either doubles the number on the board or adds 300 to it. Find the smallest possible number he can achieve after 15 minutes.

Proposed by: William Hua

Solution. $\boxed{2312}$

He doubles his initial number of 1 nine times to get 512, and then adds 300 for each remaining minute. So $512+300\cdot 6=2312$. □

10. **[6]** Triangle ABC has area 1 with $AB=AC$. Let D and E be the midpoints of AB and AC. EB and DC intersect at F. Find the area of DEF.

Proposed by: Jacob Xu

Solution. $\boxed{\frac{1}{12}}$

Note that F is the centroid, so the area of CFB is $\frac{1}{3}$. Then $\triangle FDE\sim\triangle FCB$ with ratio 2, so the answer is $\frac{1}{3}\cdot\frac{1}{4}=\boxed{\frac{1}{12}}$. □

11. **[6]** Find the number of 4-digit numbers that have at least 1 odd digit adjacent to an even digit.

Proposed by: Alexander Duncan

Solution. $\boxed{7875}$

The only numbers that don't work have digits that are either all even or all odd. Therefore the number of 4-digit numbers that satisfy this condition is $9\cdot10^3-4\cdot5^3-5^4=9000-500-625=\boxed{7875}$. □

12. **[6]** Let $P(n)$ denote the product of digits of n. Find the sum of $P(n)$ for all positive two-digit integers n.

Proposed by: James Wu

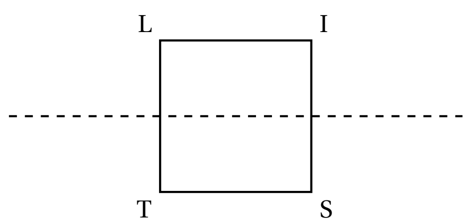
Solution. $\boxed{2025}$

For a two-digit integer $10a+b$, the product of the digits is $a\cdot b$. The sum is therefore

$$\sum_{a=1}^{9}\sum_{b=0}^{9}a\cdot b=\left(\sum_{a=1}^{9}a\right)\cdot\left(\sum_{b=0}^{9}b\right)=\left(\sum_{a=1}^{9}a\right)^2=45^2=\boxed{2025}.$$

□

13. **[6]** Square $LIST$ has side length 6. A line is drawn through the midpoints of LT and IS. Points R and H lie on this line and point O lies on side LI such that the areas of $SHORT$ and $LIST$ are equal. Find the length of RH.



Proposed by: Jacob Xu

Solution. $\boxed{9}$

Let $RH=x$. The area of $LIST$ is just $6^2=36$. Pentagon $SHORT$ is made up of triangle ROH and trapezoid $RHST$. The area of ROH is $\frac{x}{2}\cdot3$ and the area of $RHST$ is $\frac{x+6}{2}\cdot3$. Solving the equation $\frac{x}{2}\cdot3+\frac{x+6}{2}\cdot3=36$ gives $x=\boxed{9}$. □

14. **[6]** Carter starts at $(0,24)$ and heads towards $(24,6)$ in a straight line. Halfway there, however, he remembers to get milk and turns 90 degrees clockwise and moves forwards in a straight line until he reaches the y-axis. After he gets the milk, he walks to $(24,6)$ again in a straight line. Find the total distance he traveled.

Proposed by: William Hua

Solution. $\boxed{60}$

Carter travels between the following points in order:

$$(0,24)\to(12,15)\to(0,-1)\to(24,6).$$

Using the Pythagorean Theorem, his total distance traveled is $15+20+25=\boxed{60}$. □

15. **[6]** Find the sum of all distinct prime factors of 12345654321.

Proposed by: Benjamin Yin

Solution. $\boxed{71}$

Notice that $12345654321=111111^2=(111\cdot1001)^2=(3\cdot37\cdot7\cdot11\cdot13)^2$. Thus, the answer is $3+37+7+11+13=\boxed{71}$. □

16. **[6]** Find the surface area that is enclosed on a sphere of radius 10 by three arcs of length 5π which follow the curvature of the sphere.

Proposed by: Atticus Oliver

Solution. $\boxed{50\pi}$

Notice that the length of the edge is one-quarter of the circumference of the sphere. WLOG assume the first segment is placed curving from the top of the sphere to the front. It is therefore clear to see that the second and third segments must intersect either directly to the left or directly to the right. Either way, this creates an area of the sphere clearly equal to one eighth of the total surface area, or $\frac{1}{8} \cdot 4\pi(10)^2 = \boxed{50\pi}$. □

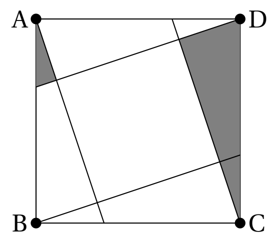
17. **[6]** Find the smallest integer $x \geq 2$ such that $3^x > x^9$.

Proposed by: Edwin Zhao

Solution. $\boxed{28}$

Let $x = 3^n$. Then, we have that $3^{3^n} > (3^n)^9$, so $3^n > 9n$. Clearly, equality occurs at $n = 3$, so equality for our original equation occurs at $x = 3^3 = 27$. Therefore, our answer is one larger than this, which is $\boxed{28}$. Alternatively, performing basic computation for each side from $x = 2$ onwards also gives us a final answer of $\boxed{28}$. □

18. **[6]** In the square below the area of the small triangle in the top left corner is 1, and the area of the larger triangle in the bottom right corner is 9. The shape in the middle is a square. Find the area of the large square.



Proposed by: Muztaba Syed

Solution. $\boxed{60}$

The idea is that the triangle on the right can be moved to the left, and the two triangles are similar with ratio 1 : 3. If we leg the longer leg of the shorter triangle be x, we see that the shorter leg of the longer triangle is also x. The longer leg of the longer triangle is $3x$, by similar triangles. The area is $3x \cdot x \cdot \frac{1}{2} = 9$, so $x^2 = 6$. The side length of the square is $x^2 + (3x)^2 = 10x^2$, so the answer is $\boxed{60}$. □

19. **[6]** A 2-digit base 10 positive integer $\overline{XY}$ is called *esab* if there exists a positive integer $b < 10$ such that the base b representation of $\overline{XY}$ is $\overline{YX}_b$. Find the sum of all 2-digit esab numbers. Express your answer in base 10.

Proposed by: Alexander Duncan

Solution. $\boxed{82}$

By definition, $\overline{XY}$ is esab if $10X + Y = bY + X$ for some $b < 10$. Rearranging gives $9X = Y(b-1)$. Since $b < 10$, neither Y nor $b-1$ may be a multiple of 9, so both Y and $b-1$ are multiples of 3. Testing out the cases where Y and $b-1$ are 3 or 6 gives 3 numbers: 13, 23, and 46. Therefore, our answer is $13 + 23 + 46 = \boxed{82}$. □

20. **[6]** For even n, define $f(n)$ to be the smallest odd prime p such that $n - p$ is not prime. Find the maximum value of $f(n)$ for all positive even $n > 10$.

Proposed by: Chris Cheng

Solution. $\boxed{7}$

For $n > 10$ the max of $f(n)$ is 7 because one of $n-3, n-5, n-7$ is divisible by 3 and is not equal to 3. For $n = 22$, $22 = 19 + 3 = 17 + 5$. So $f(22) = 7$, so the answer is $\boxed{7}$. □

21. **[6]** Every cell of a 3×3 grid can be colored black or white. For each black cell in the grid, a point is added for every adjacent black cell. Find the sum of points across all 2^9 possible grid colorings.

Proposed by: James Wu

Solution. $\boxed{3072}$

A 3×3 grid has a total of 12 adjacent pairs. Since each cell is independently colored black or white, for each pair (or edge), two cells must be black, and the remaining 7 cells can be arbitrarily colored. Each pair of adjacent black cells contributes 2 points to the sum. Therefore, the answer is

$$2^7 \times 2 \times 12 = \boxed{3072}.$$

□

22. **[6]** Find the number of ordered 2025-tuples $(a_1, a_2, \dots a_{2025})$ of non-negative integers satisfying

$$\left\lfloor \frac{a_1}{1} \right\rfloor + \left\lfloor \frac{a_2}{2} \right\rfloor + \cdots + \left\lfloor \frac{a_{2025}}{2025} \right\rfloor = 2025.$$

Proposed by: Muztaba Syed

Solution. $\boxed{\dfrac{4049!}{2024!}}$

If we know the value of $\lfloor \frac{a_k}{k} \rfloor$, there are k options for what a_k actually is. So we just need to find the number of tuples $(b_1, b_2, \dots b_{2025})$ for which $b_1 + b_2 + \cdots + b_{2025} = 2025$ and then multiply by 2025!. This can be easily computed as $\binom{4049}{2024}$ by stars and bars. The answer is then

$$\frac{4049!}{2024!2025!} \cdot 2025! = \boxed{\frac{4049!}{2024!}}.$$

□

23. **[6]** Let ABC be an equilateral triangle and $\mathscr{P}$ a plane which does not intersect ABC. Let the projections of A, B, C onto $\mathscr{P}$ be A', B', and C'. Suppose $\angle B'A'C' = 90°$ and A, B, C have heights 37, 23, and 16 to $\mathscr{P}$, respectively. Find AB^2.

Proposed by: Muztaba Syed

Solution. $\boxed{588}$

Let $A'B' = b$ and $A'C' = c$. Let $AB = s$. Then by the Pythagorean Theorem on $A'B'BA$, we have

$$(37-23)^2 + b^2 = s^2.$$

Similarly on $A'C'CA$, we get

$$(37-16)^2 + c^2 = s^2.$$

Finally do the same thing on $B'C'CB$, noting that $(B'C')^2 = a^2 + b^2$,

$$(23-16)^2 + a^2 + b^2 = s^2.$$

Combining these equations, we get

$$s^2 = 14^2 + b^2 = 21^2 + c^2 = 7^2 + b^2 + c^2.$$

This can be solved to get $b^2 = 21^2 - 7^2$, and thus $s^2 = 14^2 + b^2 = 14^2 + 21^2 - 7^2 = 49 \cdot (4 + 9 - 1) = 49 \cdot 12 = \boxed{588}$. □

24. **[6]** There exists some positive integer k such that the coefficient of x^k in the expansion of $(3x+1)^{2025}$ is maximized. Find k.

Proposed by: Peter Bai

Solution. $\boxed{1519}$

Recall that the coefficient of the x^m term in $(x+1)^n$ is equal to $\binom{n}{m}$. Since we have an extra 3, we can express the coefficient of x^k in the expansion of $(3x+1)^{2025}$ as $3^k\binom{2025}{k}$.

Now, we can calculate the ratio between the coefficients of x^{k+1} and x^k as

$$\frac{3^{k+1}\binom{2025}{k+1}}{3^k\binom{2025}{k}} = \frac{3\cdot\frac{2025!}{(k+1)!(2025-k-1)!}}{\frac{2025!}{k!(2025-k)!}} = \frac{3(k)!(2025-k)!}{(k+1)!(2025-k-1)!} = \frac{3(2025-k)}{k+1}.$$

Notice that the coefficient of x^{k+1} will be greater than the coefficient of x^k if and only if the previously calculated ratio is greater than one. This is true when

$$\frac{3(2025-k)}{k+1} > 1 \implies 6075-3k > k+1 \implies 6074 > 4k \implies 1518.5 > k$$

The smallest integer value of k that does not satisfy this inequality is $k = 1519$. Thus, the coefficients of x^k will keep increasing up until $k = \boxed{1519}$, and then they will decrease for all greater k because the ratio will be less than 1. □

25. **[6]** Let $p = 2027$ be a prime and $f(x) = x^2 - 2$. Find the remainder when

$$\underbrace{f\left(f\left(\dots f\left(\frac{p+5}{2}\right)\dots\right)\right)}_{2025\text{ times}}$$

is divided by p.

Proposed by: Muztaba Syed

Solution. $\boxed{511}$

Motivated by x^2-2, we make the substitution $x = y + \frac{1}{y}$. Indeed note that

$$f(x) = x^2 - 2 = \left(y+\frac{1}{y}\right)^2 - 2 = y^2 + \frac{1}{y^2}.$$

Thus it is easy to see by induction that

$$\underbrace{f\left(f\left(\dots f(x)\dots\right)\right)}_{n\text{ times}} = y^{2^n} + \frac{1}{y^{2^n}}.,$$

Additionally note that $\frac{5}{2} = 2 + \frac{1}{2}$. Thus we must compute

$$2^{2^{2025}} + \frac{1}{2^{2^{2025}}} \pmod{2027}.$$

To do this first compute $2^{2025} \pmod{2026}$. Modulo 2 this is 0, and modulo 1013 this is

$$2^{2025} \equiv (2^{1012})^2 \cdot 2 \equiv 1 \cdot 2 \equiv 2 \pmod{1013}.$$

Thus $2^{2025} \equiv 2 \pmod{2026}$. So we need the value of

$$4 + \frac{1}{4} \equiv 4 + \frac{2028}{4} \equiv \boxed{511} \pmod{2027}.$$

□