LMT Spring 2025 Guts Round Solutions- Part 1

Team Name:

1. **[9]** The Einstein-Pythagoras equation states that

$$E = m\left(a^2 + b^2\right).$$

If $m = \frac{1}{17}$, $a = 16$ and $b = 30$, find the value of E.

Proposed by: Peter Bai

Solution. $\boxed{68}$

Plugging in the provided information into the equation gives

$$E = \frac{1}{17}\left(16^2 + 30^2\right) = \frac{4}{17}\left(8^2 + 15^2\right) = \frac{4\cdot 17^2}{17} = 4\cdot 17 = \boxed{68}.$$

□

2. **[9]** Find the minimum possible value that can be created using the numbers $2, 0, 2$, and 5, given that only the operations addition, subtraction, multiplication, and division can be used (no concatenation).

Proposed by: Edwin Zhao

Solution. $\boxed{-20}$

$0 - 2\cdot 2\cdot 5 = \boxed{-20}$. □

3. **[9]** Find the units digit of $2^{2025} + 0^{2025} + 2^{2025} + 5^{2025}$.

Proposed by: Vedant Joshi

Solution. $\boxed{9}$

Using the following properties:

$$2^{2025} \equiv 2 \pmod{10},$$
$$0^{2025} \equiv 0 \pmod{10},$$
$$5^{2025} \equiv 5 \pmod{10},$$

we can reduce the expression to $2 + 0 + 2 + 5$ which equals 9. □

LMT Spring 2025 Guts Round Solutions- Part 2

Team Name:

4. **[10]** Drew has ten different paint colors that are split into three categories: There are three warm colors, three cool colors, and four neutral colors. Drew wants to mix two colors that are not from the same category. Find the number of ways he can choose these two colors.

Proposed by: Jacob Xu

Solution. $\boxed{33}$

The numbers of ways to choose a warm and cool color is $3\cdot 3 = 9$. The number of ways to choose a warm and neutral color is $3\cdot 4 = 12$. The number of ways to choose a cool and neutral color is $3\cdot 4 = 12$. So the total is $9 + 12 + 12 = \boxed{33}$. □

5. **[10]** Apollo rolls a standard 2-sided die, 4-sided die, and 6-sided die. Find the probability he rolls three distinct numbers.

Proposed by: Jacob Xu

Solution. $\boxed{\frac{1}{2}}$

The 4-sided die does not match the 2-sided die with probability $\frac{3}{4}$, and the 6-sided die does not match the others with probability $\frac{4}{6}$. The answer is $\frac{3}{4} \cdot \frac{4}{6} = \boxed{\frac{1}{2}}$. □

6. **[10]** Let A, B, C, D, P be points in the plane satisfying $PA = 5$, $PB = 6$, $PC = 7$, $PD = 9$. Find the maximum possible area of the quadrilateral formed by the points A, B, C, D.

Proposed by: Muztaba Syed

Solution. $\boxed{91}$

If we arrange the 4 points around P, we get 4 triangles. These areas are simultaneously maximized when $\angle P$ is $90°$. Thus the quadrilateral will have perpendicular diagonals. The diagonals sum to $5+6+7+9$, so the area will be maximized when they are close together. We can do this by making A and D opposite, giving an answer of $\frac{1}{2}(5+9)(6+7) = \boxed{91}$. □

LMT Spring 2025 Guts Round Solutions- Part 3

Team Name:

7. **[11]** Find the sum of the positive factors of 2025 that have exactly 3 positive factors.

Proposed by: Alexander Duncan

Solution. $\boxed{34}$

If a number has exactly 3 factors, then it must be the square of a prime. $2025 = 3^4 \cdot 5^2$ so the only factors of 2025 with 3 factors are 9 and 25, so our answer is $9+25 = \boxed{34}$. □

8. **[11]** There exist two positive integers a and b such that $\frac{22!\cdot 23!}{a!\cdot 24!} = \binom{b}{4}$. Find $a+b$.

Proposed by: Jacob Xu

Solution. $\boxed{40}$

$\frac{22!\cdot 23!}{a!\cdot 24!} = \frac{22!}{a!\cdot 24} = \frac{22!}{a!\cdot 4!}$. We observe that $a = 18$ and $b = 22$ makes the equation true. So the answer is $18+22 = \boxed{40}$. □

9. **[11]** Define a sequence of numbers $n_1, n_2, \ldots$ such that $n_i n_{i+1} = n_{i+2}$ for all positive integers i. Given that $n_1 = n_{2026}$ and $n_2 = n_{2027}$, find the number of possible ordered tuples $(n_1, n_2 \ldots n_{2024}, n_{2025})$.

Proposed by: Edwin Zhao

Solution. $\boxed{5}$

Notice that all the ns must be either one, zero, or negative one. Otherwise, the sequence would keep on growing (or getting closer to zero). Our trivial solutions are when they are all 0 or they are all 1. Alternatively, you could have them be $(-1,-1,1,-1,-1,1\ldots)$, and so on. However, notice that there are three options for this, as it could also be shifted to the left by 1 or to the right by 1. Thus, our final answer is $1+1+3 = \boxed{5}$] □

______ 10. **[12]** Find

$$\sum_{a\ge 0}\sum_{b\ge 0}\frac{1}{2^a3^b5^{a+b}}.$$

Proposed by: Samuel Tsui

Solution. $\boxed{\frac{25}{21}}$

Note that sum is $\sum_{a\ge 0}\sum_{b\ge 0}\frac{1}{2^a3^b5^{a+b}}=\sum_{a\ge 0}\sum_{b\ge 0}\frac{1}{10^a15^b}=\frac{1}{1-\frac{1}{10}}\cdot\frac{1}{1-\frac{1}{15}}=\boxed{\frac{25}{21}}$. □

______ 11. **[12]** Suppose a,b,c are the roots of x^3-ax^2-bx-c. Find all possible values of c.

Proposed by: Samuel Tsui

Solution. $\boxed{0,-1}$

Plugging a into the polynomial gives $ab+c=0$. From Vieta's $a+b+c=a$ which means $c=-b$. Substituting this into the equation we get that either $b=0$ or $a=1$. When $b=0$, $c=-b=0$ and when $a=1$, Vieta's yields $abc=c$ so $b=1$ and $c=-b=-1$. Thus the answer is $\boxed{0,-1}$. □

______ 12. **[12]** Call a number n *factormaxxing* if exactly 4 of its factors are in the set $\{2,3,4,5,8,9,16,25,27,32,81,125,243,625,3125\}$. Given that n has no prime factors other than $2,3$, and 5, find the number of possible values of n that are *factormaxxing*.

Proposed by: Edwin Zhao

Solution. $\boxed{15}$

Notice that all numbers n can be generated by the following factorization: $(1+2+4+8+16)(1+3+9+27+81)(1+5+25+125+625)$. We can then use stars and bars to get $\binom{6}{2}=\boxed{15}$. □

LMT Spring 2025 Guts Round Solutions- Part 5

Team Name: ______________

______________ 13. **[13]** There are 46 freshmen and 35 sophomores who place themselves in distinct cells of a 9×9 grid. In a move, two people can switch places (these people don't need to be adjacent). Sophomores all hate each other, so they don't want to be in adjacent cells. Find the smallest k for which we can always perform at most k moves to make sure no two sophomores are adjacent.

Proposed by: Muztaba Syed

Solution. $\boxed{17}$

If we checkerboard color the grid, note that we can put all the sophomores on the same color cells to complete the task. If there are a sophomores on one color, then there are $35-a$ sophomores on the other color. So we can always do this in $\min(a, 35-a) \leqslant 17$ moves.

To show that we can't do better than this, split them into 17 pairs of adjacent people and place the 35th person arbitrarily. Then in each pair at least one person needs to be moved, so we need at least $\boxed{17}$ moves in this arrangement. □

______________ 14. **[13]** A function $g(t) = 2t^3 + 9$ satisfies $g(x) - g(y) = 126$ for some positive numbers x and y. If $x - y = 3$, find the value of $x + y$.

Proposed by: Jonathan Liu

Solution. $\boxed{5}$

$g(x) - g(y) = 2(x^3 - y^3) = 126$ Using difference of cubes, $(x-y)(x^2+xy+y^2) = 63$. Since $x - y = 3$, we know $x^2 + xy + y^2 = 21$. Since $x - y = 3$, we know $x^2 - 2xy + y^2 = 9$ and $xy = 4$. We find that $x^2 + 2xy + y^2 = (x+y)^2 = 25$, and since x and y are positive, $x + y = \boxed{5}$ □

______________ 15. **[13]** Four spheres of radius 6 are mutually externally tangent. Point P is chosen such that it is d units from the center of each sphere. Find d.

Proposed by: Atticus Oliver

Solution. $\boxed{3\sqrt{6}}$

Constructing the segments that join the centers of the spheres gives a regular tetrahedron with side length 12. P is clearly the circumcenter of this tetrahedron, so constructing the four circumradii of the tetrahedron gives four smaller congruent tetrahedra. These tetrahedra therefore have a height that is $\frac{1}{4}$ of the total height of the tetrahedron, so the length of the circumradius is therefore $\frac{3}{4}$ of the total height of the tetrahedron. This height can be calculated with the Pythagorean Theorem as $\sqrt{12^2 - 6^2 - (2\sqrt{3})^2} = 4\sqrt{6}$, giving d as $\boxed{3\sqrt{6}}$. □

LMT Spring 2025 Guts Round Solutions- Part 6

Team Name: ______________

______________ 16. **[14]** Tiger is taking Day 1 of the 2025 IMO TST. The test consists of three problems. Tiger's solution to each problem obtains a score that is chosen uniformly at random from $[0,1]$. However, Tiger accidentally misorders his solutions! For problem 1, he submitted the solution that got the highest score, for problem 2, he submitted the solution that got the second highest score, and for problem 3, he submitted the solution that got the lowest score. Any solution corresponding to the right problem earns a score equal to the solution's score, while any problem in the wrong position gets a score of 0. Find the expected value of Tiger's score.

Proposed by: Evin Liang

Solution. $\boxed{\frac{1}{2}}$

By linearity of expectation, the expected total score is the sum of the expected scores for each problem. Let the scores Tiger gets for each problem be a, b, and c. Because there are 3 problems in total, any solution has a $\frac{1}{3}$ chance of corresponding to the right problem. Thus, the expected value of Tiger's score is $\frac{1}{3}E_a + \frac{1}{3}E_b + \frac{1}{3}E_c$, where E_a, E_b, E_c denote the expected values of a, b, and c respectively. Since his scores are uniformly distributed in $[0,1]$ and he puts them in decreasing order, the expected values of a, b, c are $\frac{3}{4}, \frac{1}{2}, \frac{1}{4}$. Thus, our answer is $\frac{1}{3} \cdot \frac{3}{4} + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{4} = \boxed{\frac{1}{2}}$. □

__________ 17. **[14]** Each second, ST rolls a fair 6-sided die and records the result. If the sum of the numbers ST has written ever exceeds 9, he will erase the last number he wrote. Find the expected number of times he needs to roll the dice before the sum is exactly 9.

Proposed by: Alexander Duncan

Solution. $\boxed{\frac{265}{36}}$

Any sum from 3 to 8 is identical as there is a $\frac{1}{6}$ chance the sum will become 9 and a $\frac{5}{6}$ chance the sum will become a number from 3 to 8. Since there is a $\frac{1}{6}$ chance the sum will become 9, the expected number of times ST has to roll once he first gets a number between 3 and 8 is 6. There is $\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$ chance that he rolls 2 ones, taking 3 turns to get into the range. There is a $\frac{5}{6} \cdot \frac{1}{6} + \frac{1}{6} = \frac{11}{36}$ chance that he either rolls a one and a number greater than one or that he rolls a two, taking 2 turns. There is a $\frac{2}{3}$ chance that he rolls a number greater than two first, taking 1 turn. In total, the expected number of rolls is $3 \cdot \frac{1}{36} + 2 \cdot \frac{11}{36} + 1 \cdot \frac{2}{3} + 6 = \boxed{\frac{265}{36}}$. □

__________ 18. **[14]** Semicircle O has diameter $AB = 12$. Arc $BC = 135°$. Let D be the midpoint of arc $\widehat{BC}$. Find the area of the region bounded by the lines AD, CD and arc $\widehat{AC}$.

Proposed by: Jonathan Liu

Solution. $\boxed{\frac{9\pi}{2}}$

Notice that lines AC and OD are parallel, since AC makes a $\frac{180-45}{2} = \frac{135}{2}$ degree angle with AB and OD also makes a $\frac{135}{2}$ degree angle with AB, since it is the midpoint of arc BC.

Therefore, $[ACD] = [AOC]$ because base x height is constant. Finally, notice our area requested is just the area of sector AOC. This is $\frac{1}{8}$th of the entire circle, which is $\frac{1}{8} \cdot 36\pi = \boxed{\frac{9\pi}{2}}$. □

LMT Spring 2025 Guts Round Solutions- Part 7

Team Name:

__________ 19. **[15]** Let point P lie inside isosceles right triangle ABC with $AB = BC$. Suppose P has distances 3, 4, and 5 to sides AB, BC, and CA, respectively. Find AB.

Proposed by: Muztaba Syed

Solution. $\boxed{7+5\sqrt{2}}$

Let $AB = x$. We can calculate the area of ABC in two ways. First, we can split it into triangles ABP, BCP, and CAP. This gives us

$$\frac{3x}{2} + \frac{4x}{2} + \frac{5\sqrt{2}x}{2}.$$

But the area is also $\frac{x^2}{2}$, so we get

$$\frac{3x}{2}+\frac{4x}{2}+\frac{5\sqrt{2}x}{2}=\frac{x^2}{2} \implies x=\boxed{7+5\sqrt{2}}.$$

$\square$

_____________ 20. **[15]** Find the number of nonnegative integers (a,b,c,d) such that

$$\begin{aligned} a+b &\le 4, \\ a+d &\le 4, \\ c+b &\le 4, \\ c+d &\le 4. \end{aligned}$$

Proposed by: Jonathan Liu

Solution. $\boxed{155}$

Notice $b \le 4-a$ and $b \le 4-c$. Also $d \le 4-a$ and $d \le 4-c$. Thus $b,d \le 4-\max(a,c)$

If $\max(a,c)=n$, then there are $2n+1$ ordered pairs (a,c) that satisfy $\max(a,c)=n$, since there are no constraints on $a+c$.

Next, note that there's $(5-n)^2$ pairs of (b,d) that satisfy $b,d \le 4-\max(a,c)$, since each of b and d range from 0 to $4-n$, giving $5-n$ options for each of b and d.

Thus, the answer is $\sum_{n=0}^{4}(2n+1)(5-n)^2=\boxed{155}$. $\square$

_____________ 21. **[15]** Find the number of ordered quadruples of nonnegative integers w,x,y,z such that

$$\left\lfloor\frac{2025}{10^k}\right\rfloor=\left\lfloor\frac{w}{10^k}\right\rfloor+\left\lfloor\frac{x}{10^k}\right\rfloor+\left\lfloor\frac{y}{10^k}\right\rfloor+\left\lfloor\frac{z}{10^k}\right\rfloor$$

holds for all nonnegative integers k.

Proposed by: Micah Wang

Solution. $\boxed{5600}$

We let $w=\overline{w_1w_2w_3w_4}$, and define x, y, and z similarly. The $k=0$ case clearly implies that $2025=w+x+y+z$. For $k \ge 4$, the equation clearly becomes $0=0$ for $w,x,y,z<10000$, which is trivial. Notice that at $k=3$ we have $\lfloor 2.025\rfloor=\lfloor\overline{w_1.w_2w_3w_4}\rfloor+\lfloor\overline{x_1.x_2x_3x_4}\rfloor+\lfloor\overline{y_1.y_2y_3y_4}\rfloor+\lfloor\overline{z_1.z_2z_3z_4}\rfloor$, which simplifies to $2=w_1+x_1+y_1+z_1$. Therefore the thousands digits of w, x, y, and z sum to the thousands digit of 2025, so there is no carry from the hundreds digit. The same logic can be applied to prove the tens digit does not carry when $k=2$ and that the ones digit does not carry when $k=1$. Therefore $w_1+x_1+y_1+z_1=2$, $w_2+x_2+y_2+z_2=0$, $w_3+x_3+y_3+z_3=2$, and $w_4+x_4+y_4+z_4=5$. Each of these cases can be solved with stars and bars to give $\binom{5}{3}$ choices for the first digits, $\binom{3}{3}$ choices for the second digits, $\binom{5}{3}$ choices for the third digits, and $\binom{8}{3}$ choices for the last digits, for a total of $10\cdot 1\cdot 10\cdot 56=\boxed{5600}$ cases. $\square$

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LMT Spring 2025 Guts Round Solutions- Part 8

Team Name:

22. **[17]** Find the number of three digit perfect squares $\underline{a}\,\underline{b}\,\underline{c}$ for which $a(2025^2)+b(2025)+c$ is also a perfect square.

Proposed by: Muztaba Syed

Solution. $\boxed{9}$

In base k, we can sometimes factor ak^2+bk+c as the square of a binomial. With some wishful thinking, it is unlikely that this will be a perfect square when $k=2025$ if it does not factor in this way. This can be verified by bashing.

Thus the possible factorizations are (noting that a, b, c cannot be larger than 10)

$$k^2, (k+1)^2, (k+2)^2, (k+3)^2, (2k)^2, (2k+1)^2, (2k+2)^2, (3k)^2, (3k+1)^2.$$

So the answer is $\boxed{9}$. □

23. **[17]** Find the number of positive integers c less than 2025 such that there exist positive integers a and b satisfying

$$2^a+2^b=c^2.$$

Proposed by: Chris Cheng, Samuel Tsui

Solution. $\boxed{20}$

Letting $a=b$ we find that all powers of 2 greater than or equal to 2 work. Otherwise WLOG $a>b$ and divide by 2^b (note b must be even). We have $2^{a-b}+1=d^2$ so $2^{a-b}=(d-1)(d+1)$ meaning $d=3$ so all numbers which are three times a power of 2 also work. Extracting gives the answer is $\boxed{20}$. □

24. **[17]** Sam and Jonathan take turns flipping a fair coin, and they stop when they collectively flip three heads in a row. Given Sam flips first, find the probability he flips the last head.

Proposed by: Samuel Tsui

Solution. $\boxed{\dfrac{6}{11}}$

Let p be the answer. Note if Sam first flips tails or they both flip heads then he flips tails the probability he flips the last head is now $1-p$. Moreover, if Sam flips heads and Jonathan flips tails the probability stays the same. Thus we get $p=(\frac{1}{2}+\frac{1}{8})(1-p)+\frac{1}{4}p+\frac{1}{8}$ and solving gives $p=\boxed{\dfrac{6}{11}}$. □

LMT Spring 2025 Guts Round Solutions- Part 9

Team Name:

25. **[20]** Let ABC be an equilateral triangle with side length 1. Let Ω be the circumcircle of triangle ABC, and let P lie on minor arc AB of Ω. The tangent line at P intersects AB at D and AC at E such that $PD=PE$. Find the area of ADE.

Proposed by: Samuel Tsui

Solution. $\boxed{\frac{\sqrt{3}}{4}}$

Complete equilateral triangle DEF such that (ABC) is the incircle. This has to exist since P is the midpoint of DE while DE is tangent to (PE) at D. Thus, if O is the center of (ABC), we have $OP = \frac{AC}{\sqrt{3}}$, but since DEF is equilateral, we have $DP = \sqrt{3}OP = AC$. Then, by POP, $AC^2 = EA \cdot (EC) = EA \cdot (EA + AC)$, so $EA = \frac{\sqrt{5}-1}{2} \cdot AC$. To finish, note that

$$[ADE] = \frac{1}{2}\sin 120^\circ AD \cdot AE = \frac{1}{2}\sin 120^\circ \frac{\sqrt{5}-1}{2}\frac{\sqrt{5}+1}{2} \cdot AC^2 = \frac{\sqrt{3}}{4} \cdot AC^2.$$

So the answer is $\boxed{\frac{\sqrt{3}}{4}}$. □

__________ 26. **[20]** Let a pair of integers (x, y) be called *yearly* if

$$(x+y)^2 = 100x + y.$$

For example, $(20, 25)$ is a yearly pair. There exists two yearly pairs $(x_1, y), (x_2, y)$ such that $y < -100$ is maximized. Given that $x_2 > x_1$, find (x_2, y).

Proposed by: Ryan Tang

Solution. $\boxed{(394, -196)}$

Expand, and we have

$$x^2 + (2y-100)x + y^2 - y = 0,$$

and by quadratic formula,

$$x = 50 - y \pm \sqrt{y^2 - 100y + 50^2 - y^2 + y} = 50 - y \pm \sqrt{50^2 - 99y}$$

Thus, we want to find all perfect squares such that $k^2 \equiv 50^2 \equiv 25 \pmod{99}$. By CRT, $k^2 \equiv 7 \pmod 9$, $k^2 \equiv 3 \pmod{11}$. The first equation has solutions at $k \equiv 4, 5 \pmod 9$ and the second has solutions at $k \equiv 5, 6 \pmod{11}$. By CRT, $k \equiv 5, 49, 50, 94 \pmod{99}$. Now, note that $k^2 \geq 50^2 - (99 \cdot 100) = 12400$, so $k \geq 10\sqrt{124} > 110$. $k = 104$ doesn't work, so $k = 49 + 99 = 148$ is the next possible value. Thus, $y = \frac{50^2 - 148^2}{99} = -\frac{198 \cdot 98}{99} = -196$. This gives $x = 50 + 196 \pm 148$. This gives the answer $x = 394$. □

__________ 27. **[20]** Let $f(x)$ be a quartic polynomial with roots w, x, y, z such that

$$w = \frac{1}{5 - xyz}, \qquad x = \frac{1}{7 - wyz}, \qquad y = \frac{1}{9 - wxz}, \qquad z = \frac{1}{11 - wxy}$$

and $f(0) = 3$. Find $f(1 + wxyz)$.

Proposed by: Ryan Tang

Solution. $\boxed{5760}$

Let $f(r) = a(r-w)(r-x)(r-y)(r-z)$. We have $awxyz = 3$ from the given condition.

Rearrangement implies $1 + wxyz = 5w = 7x = 9y = 11z$. Hence,

$$\begin{aligned} f(1+wxyz) &= a(wxyz+1-w)(wxyz+1-x)(wxyz+1-y)(wxyz+1-z) \\ &= a(5w-w)(7x-x)(9y-y)(11z-z) \\ &= 4 \cdot 6 \cdot 8 \cdot 10 \cdot awxyz \\ &= \boxed{5760}. \end{aligned}$$

□

____________ 28. **[23]** Let $ABCD$ be a parallelogram with $AB = 20$, $BC = 25$. Let E be the midpoint of BC, and let F be the foot of the altitude from D to AE. Given that $AF = 15$, find the area of triangle CDF.

Proposed by: Henry Eriksson

Solution. $\boxed{100\sqrt{3}}$

We will start by showing that $CF = CD = 20$. Let P be the intersection of the extensions of lines CD and AE. $CE \parallel AD$ and $CE = \frac{1}{2}AD$, so CE is the midline of ADP. This means that C is the midpoint of DP. The triangle DFP is a right triangle, so the length of the median (which is CF) is half the length of the hypotenuse (which is $\frac{1}{2}DP = CD$).

By the Pythagorean Theorem, $DF^2 = AD^2 - AF^2 = 25^2 - 15^2 = 40 \cdot 10 = 400 \to DF = 20$. But also $CD = 20$ and $CF = 20$, so triangle CDF is equilateral. This means that its area is $\frac{\sqrt{3}}{4} \cdot 20^2 = \boxed{100\sqrt{3}}$. □

____________ 29. **[23]** Tiger is taking Day 2 of the 2025 IMO TST. The test consists of three problems. This time, Tiger decides to not look at the problem numbers. When doing the nth problem, he gives the problem a difficulty rating, chosen uniformly at random from $[0, n]$. He then orders the problems in increasing order of difficulty. Find the probability that none of the problems in Tiger's ordering are in the same place they were in for the original ordering.

Proposed by: Evin Liang

Solution. $\boxed{\dfrac{5}{36}}$

There are two cases. Either the ordering is $2 < 3 < 1$ or the ordering is $3 < 1 < 2$.

If the ordering is $2 < 3 < 1$, then Tiger's ratings for p2 and p3 must both be in $[0, 1]$, which happens with probability $\frac{1}{2} \cdot \frac{1}{3}$. Then all of the 3! possible orderings of the ratings are equally likely, so the probability the ordering is $2 < 3 < 1$ is $\frac{1}{3!}$. So the overall probability for this case is $\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3!} = \frac{1}{36}$.

If the ordering is $3 < 1 < 2$, then Tiger's rating for p3 must be in $[0, 1]$, which has probability $\frac{1}{3}$. Then we split cases based on whether Tiger's rating for p2 is greater than 1 or less. Tiger's p2 rating is greater than 1 with probability $\frac{1}{2}$, and then the probability that Tiger's p3 rating is less than the p1 rating is $\frac{1}{2}$. Tiger's p2 rating is less than 1 with a probability of $\frac{1}{2}$, and then the probability that the ratings for the three problems are in the right order is $\frac{1}{6}$. Hence the overall probability for this case is $\frac{1}{3}\left(\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{6}\right) = \frac{1}{9}$.

Thus, the total probability is $\frac{1}{36} + \frac{1}{9} = \boxed{\dfrac{5}{36}}$. □

____________ 30. **[23]** Let ABC be a triangle such that $AB = 3$, $AC = 4$, $BC = 5$. A circle through A intersects AB at P, AC at Q, and BC at R and S such that $\angle PQS = \angle QPR = 60^\circ$. Find $\frac{BR}{CR} + \frac{BS}{CS}$.

Proposed by: Samuel Tsui

Solution. $\boxed{\sqrt{3}}$

Note $\angle BAS = \angle PAS = \angle PQS = 60^\circ$ and $\angle CAR = \angle QAR = \angle QPR = 60^\circ$ so R and S are the feet of the angle trisectors of $\angle BAC$. Thus by ratio lemma $\frac{BR}{CR} + \frac{BS}{CS} = \frac{3}{4}\left(\frac{\sin 30^\circ}{\sin 60^\circ} + \frac{\sin 60^\circ}{\sin 30^\circ}\right) = \boxed{\sqrt{3}}$. □

31. **[26]** Two 10 by 10 grids with squares colored red and blue are called *evil* if for every row, column, or diagonal, the number of red cells is the same for both. Find the maximum number of red cells in common between two distinct 10 by 10 grids that are evil.

Proposed by: Evin Liang

Solution. $\boxed{92}$

Consider the set of squares that are different between the two grids. This must be nonempty and intersect every row, column, or long diagonal in an even number of squares. If we assign coordinates, this means that the number of cells of maximum x, $-x$, y, $-y$, $x+y$, $-x-y$, $x-y$, or $y-x$ must be even. So the convex hull of the set must have at least 8 sides and hence 8 vertices. (clunky formalization but the main idea is not scary) Thus the answer is at most 92. This is achieved by an exercise to the reader. □

32. **[26]** Let $ABCD$ be a cyclic quadrilateral. Given that $AB = CD = 2$ and the distance between any two points in the set $\{A,B,C,D\}$ is a positive integer, find the sum of all possible perimeters of $ABCD$.

Proposed by: Edwin Zhao

Solution. $\boxed{11}$

Let $BC = n$, $AD = p$, and $AC = q$. Since $AB = CD$, $AB = CD$ so $\angle ABC$ and $\angle BCD$ each inscribe an arc of the same length and they are equal. Then, by SAS congruence, $\triangle ABC$ is congruent to $\triangle DBC$. Therefore, $AC = BC = q$. WLOG let $BC < AD$. By Ptolemy's, we have $4 + np = q^2$. Clearly, $AB + BC + CD > AD \implies AD - BC < 4 \implies p - n < 4$, as otherwise the quadrilateral would be degenerate. We will now casework on $p - n$. If $p - n = 0$, then $ABCD$ is just a rectangle and q is a hypotenuse of a right triangle with legs 2 and n. However, there are no integer length right triangles with one of its legs being 2, so there are no possibilities in this case. If $p - n = 1$, then upon substituting, we get $n^2 + n + 4 = q^2$. Let q be $n + a$ for positive integer a. Then, we have $n + 4 = 2an + a^2$ and $(2a-1)n = 4 - a^2$. The only way both sides are positive is if $a = 1$. This gives $n = 3$, so $p = 4$ and our perimeter is $2+2+3+4 = 11$ for this case. If $p-n = 2$, we have $n^2+2n+4 = q^2$. We then have $2n + 4 = 2an + a^2$ and $(2a-2)n = 4 - a^2$, and there are no possible values of a in this case that make both sides positive. If $p - n = 3$, then we have $n^2 + 3n + 4 = q^2$ and $(2a-3)n = 4 - a^2$. Again, there are no solutions, so the only possibility for the perimeter of $ABCD$ is $\boxed{11}$. □

33. **[26]** Let $S = \{1,2,3,4\}$. Compute the number of functions $f : S \to S$ such that no $x \in S$ satisfy $f(f(x)) + x = 2f(x)$.

Proposed by: Adam Ge, Samuel Tsui, Edwin Zhao

Solution. $\boxed{52}$

The condition is satisfied when

- $f(a) = a$ for some $a \in S$, or
- $f(a) = b, f(b) = c$ for distinct $a,b,c \in S$ that form an arithmetic progression.

From the first bullet, $f(a) \neq a$ for all $a \in S$. Thus, ignoring the second bullet point, there would be $3^4 = 81$ possible functions. We can use complementary counting and subtract the functions that satisfy both bullets from 81.

Observe that there are only two arithmetic sequences in S, namely $\{1,2,3\}$ and $\{2,3,4\}$. If there is exactly 1 triple (a,b,c) that satisfies the second bullet, WLOG say $\{1,2,3\}$, then either $f(1) = 2$ and $f(2) = 3$ or $f(3) = 2$ and $f(2) = 1$.

If $f(1) = 2$ and $f(2) = 3$, then $f(3)$ can be 1 or 2. If $f(3) = 1$, $f(4)$ can be 1, 2, or 3 but if $f(3) = 2$ then $f(4)$ can only be 1 or 2. This yields 5 functions.

If $f(3) = 2$ and $f(2) = 1$, then $f(1)$ can be 2, 3, or 4 and $f(4)$ can be 2 or 1. This yields 6 functions.

Thus, there are 11 functions when $\{1,2,3\}$ is the only arithmetic sequence satisfying the second bullet, which we multiply by 2 to account for $\{4,5,6\}$.

Now consider when there are two triples that satisfies the second bullet; these triples must be $\{1,2,3\}$ and $\{2,3,4\}$. Observe that $f(3) = 2$ or $f(3) = 4$, and $f(2) = 1$ or $f(2) = 3$, giving $2 \times 2 = 4$ cases:

- If $f(3) = 2$ and $f(2) = 1$, $f(4) = 3$ and $f(1)$ can be 2, 3, or 4, yielding 3 functions.
- If $f(3) = 2$ and $f(2) = 3$, then $f(4) = 3$ and $f(1) = 2$, yielding 1 function.
- If $f(3) = 4$ and $f(2) = 1$, it is impossible to obtain both arithmetic sequences, yielding 0 functions.
- If $f(3) = 4$ and $f(2) = 3$, $f(1) = 2$ and $f(4)$ can be 1, 2, or 3, yielding 3 functions.

There are 22 functions if only 1 arithmetic sequence satisfies the second bullet and 7 functions if both arithmetic sequences satisfy the second bullet, so the answer is $81-(7+22) = 52$. □

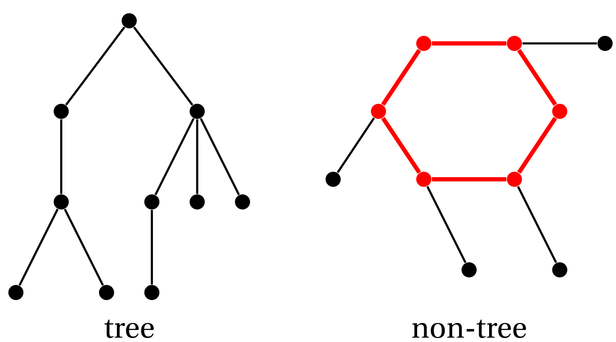
LMT Spring 2025 Guts Round Solutions- Part 12

Team Name:

34. **[30]** Let G be a tree chosen uniformly at random from the set of unlabeled trees with 10 vertices. For an ordered pair of vertices $V, W \in G$, let $d(V,W)$ be the fewest number of edges needed to walk from vertex V to vertex W along G. Estimate the expected value of

$$\sum_{V,W\in G} d(V,W).$$

If your answer is A and the correct answer is E, you will receive $\lfloor \max\left(0, 20 - 2\lfloor |E-A| \rfloor^{0.5}\right) \rfloor$ points. *Note:* a *tree* is a connected graph with no cycles.



Proposed by: Benjamin Yin and Jerry Xu

Solution. $\boxed{246.0566}$

Minimum $\sum_{V,W\in G} d(V,W)$ is a graph with 9 vertices connect to a root vertex, giving $2\left[2\cdot\binom{9}{2}+9\right] = 162$. Maximum $\sum_{V,W\in G} d(V,W)$ is a graph with 10 vertices connected in a row, giving $2\left[45+36+28+\cdots+1\right] = 330$. Averaging these out we get $\frac{162+330}{2} = 246$, rewarding full points. □

35. **[30]** A positive integer is *beautiful* if it can be expressed as $a^k + b^k$ for positive integers a, b, and k with $k > 1$. A positive integer is a *peacock* if it is ten digits and uses every digit from 0 to 9 exactly once. Estimate the number of beautiful peacocks. If A is the correct answer and E is your estimate, you will receive $\left\lfloor 20\min\left(\frac{A}{E}, \frac{E}{A}\right)^3 \right\rfloor$.

Proposed by: Benjamin Yin

Solution. $\boxed{570474}$

Program □

36. **[30]** Let P_n denote the polynomial with integer coefficients such that

$$P_n(\cos\theta)\sin\theta = \sin((n+1)\theta)$$

for all real θ. Estimate

$$\sum_{i=1}^{\infty} \frac{P_i(2)}{P_{2i}(2)}.$$

If your answer is A and the correct answer is E, you will be awarded $\left\lfloor 20 \min\left(\frac{A}{E}, \frac{E}{A}\right)^2 \right\rfloor$ points.

Proposed by: Jerry Xu

Solution. $\boxed{0.308}$

We have $P_n(\cosh\theta)\sinh\theta = \sinh((n+1)\theta)$. Hence,

$$\sum_{i=1}^{\infty} \frac{P_i(\cosh\theta)}{P_{2i}(\cosh\theta)} = \sum_{i=1}^{\infty} \frac{\sinh((i+1)\theta)}{\sinh((2i+1)\theta)} = \sum_{i=1}^{\infty} \frac{e^{(i+1)\theta} - e^{-(i+1)\theta}}{e^{(2i+1)\theta} - e^{-(2i+1)\theta}}$$

Because the subtracted terms get very small we ignore them and we get

$$\sum_{i=1}^{\infty} \frac{P_i(\cosh\theta)}{P_{2i}(\cosh\theta)} \approx \sum_{i=1}^{\infty} e^{-i\theta} = \frac{e^{-\theta}}{1-e^{-\theta}}.$$

Now note that if $\cosh\theta = 2$, then $e^{\theta} + e^{-\theta} = 4$ so $e^{\theta} = 2 \pm \sqrt{3}$. We choose the bigger root because it makes the error smaller, so $\sum_{i=1}^{\infty} \frac{P_i(\cosh\theta)}{P_{2i}(\cosh\theta)} \approx \frac{2-\sqrt{3}}{-1+\sqrt{3}} \approx 0.366$ (the actual value is 0.308) □