Accuracy Round Solutions

LMT Spring 2025

May 3, 2025

1. **[6]** Find
$$\text{lcm}(2, 025) + \text{lcm}(20, 25) + \text{lcm}(202, 5).$$

   Proposed by: Atticus Oliver

   Solution. $\boxed{1160}$

   $\text{lcm}(2, 025) = 50$, $\text{lcm}(20, 25) = 100$, and $\text{lcm}(202, 5) = 1010$. So the answer is $50 + 100 + 1010 = \boxed{1160}$. □

2. **[7]** Ben has 25 total pennies, nickles, and dimes. He has at least 14 pennies and nickles, at least 14 nickles and dimes, and at least 14 pennies and dimes. Find the maximum possible total value of Ben's coins, in cents.

   Proposed by: Muztaba Syed

   Solution. $\boxed{168}$

   Ben cannot have more than 11 of each coin, or the combined total number of other coins will be less than 14. To maximize the value, we give Ben 11 dimes, and then 11 nickels with 3 pennies remaining. So the answer is $11 \cdot 10 + 11 \cdot 5 + 3 = \boxed{168}$. □

3. **[7]** Equilateral triangle ABC and square $ADEF$ both have side length 2 and have a common vertex A. Find the maximum possible area of BCE.

   Proposed by: Jacob Xu

   Solution. $\boxed{\sqrt{3} + 2\sqrt{2}}$

   The area is maximized when E, A, and the midpoint of BC are collinear since the height from E to BC will be maximized. The height is equal to the sum of the height of ABC and the diagonal of $ADEF$ which is $\sqrt{3} + 2\sqrt{2}$. So the area is $2 \cdot (\sqrt{3} + 2\sqrt{2}) \cdot \frac{1}{2} = \boxed{\sqrt{3} + 2\sqrt{2}}$. □

4. **[8]** Find the number of ways to arrange the numbers $n^1, n^2, n^3, \ldots, n^{2025}$, such that there exists a real number n for which the list is in strictly increasing order.

   Proposed by: Atticus Oliver

   Solution. $\boxed{4}$

   The $n = 1$, $n = 0$, and $n = -1$ cases can immediately be discarded since multiple elements have the same value. Now notice that $\left|n^{k+1}\right| > \left|n^k\right|$ for $|k| > 1$, with the opposite inequality holding for $|k| < 1$. Further notice that the sign of n^k flips as k increases only if n is negative. Each of these cases provides two independent possibilities for the sequence, for a total of $\boxed{4}$ cases. □

5. **[8]** At a party with 2025 families, one-third have exactly one child, one-third have exactly two children, and one-third have exactly three children. A child is selected uniformly at random from all children at the party. Find the expected number of siblings they have.

   Proposed by: Samuel Tsui

Solution. $\boxed{\dfrac{4}{3}}$

The ratio of children from one child families, two child families, and three child families is 1 : 2 : 3. So of all the children, $\frac{1}{6}$ have no siblings, $\frac{2}{6}$ have one sibling, and $\frac{3}{6}$ have two siblings. Thus the answer is $\frac{1}{6}\cdot 0+\frac{2}{6}\cdot 1+\frac{3}{6}\cdot 2=\boxed{\dfrac{4}{3}}$. □

6. **[9]** A word is *admitting* if it shares more letters with the word “orz” than the word “zro”. Find the number of admitting 3-letter words that don’t contain any letters other than “o”, “r”, and “z”. A letter is shared if both words have the same letter in the same position.

   Proposed by: Alexander Duncan

   Solution. $\boxed{9}$

   A word will share an equal amount of letters with "orz" and "zro" if and only if the first and third letters are the same, so there are $3\cdot 3=9$ words that share an equal amount of letters with "orz" and "zro". There are $3^3=27$ total words so 18 of them don’t share an equal amount of letters with "orz" and "zro". Of these, half of them will be admitting, so the answer is $\boxed{9}$. □

7. **[9]** Let $ABCD$ be a trapezoid with $AB \parallel CD$, $AB \perp BC$, and $\frac{CD}{AB}=2$. Let AC and BD intersect at X. Suppose triangle BCX has area 1. Find the area of pentagon $ABCDX$.

   Proposed by: Alexander Duncan

   Solution. $\boxed{\dfrac{7}{2}}$

   Triangle ABX is similar to triangle CDX, so $[CDX]=4[ABX]$. Also, $[BCD]=2[ABC]$, so $1+4[ABX]=2(1+[ABX])$. Solving gives $[ABX]=\frac{1}{2}$, so $[CDX]=2$. Therefore the answer is $1+\frac{1}{2}+2=\boxed{\dfrac{7}{2}}$. □

8. **[10]** Suppose N is a positive integer for which $5N^2$ has exactly 18 factors, and $7N^2$ has exactly 20 factors. Find the value of N.

   Proposed by: Rohan Danda

   Solution. $\boxed{175}$

   Assume $v_p(N)=a$ (multiplicity of N, for example, $v_2(2^n)=n$), where p is not 5 or 7. Then the contribution of p to the number of divisors is $(2a+1)\mid 18$, $2a+1\mid 20$. This means $a=0$.

   Thus we can set $N=5^a7^b$. Then, $(a+1)(2b+1)=9,(2a+1)(b+1)=10$. We have $2\nmid 2a+1\mid 10$, so $2a+1=5$. Thus, $a=2$, $b=1$. The answer is then $5^2\cdot 7=\boxed{175}$. □

9. **[11]** There are 16 distinct expressions that can be created by filling in each blank of

   $$2\_\_2\_\_2\_\_5\_\_5$$

   with a × or ÷ symbol. Find the sum of all 16 expressions.

   Proposed by: Jacob Xu

   Solution. $\boxed{338}$

   Each number is either divided or multiplied, so we can write the sum of these 16 expressions in this factored form:

   $$2\left(2+\frac{1}{2}\right)\left(2+\frac{1}{2}\right)\left(5+\frac{1}{5}\right)\left(5+\frac{1}{5}\right)=2\cdot\frac{5}{2}\cdot\frac{5}{2}\cdot\frac{26}{5}\cdot\frac{26}{5}=2\cdot 13\cdot 13=\boxed{338}.$$

   To show this is equivalent, one can expand the expression to see all 16 distinct expressions included in the sum. □

10. **[11]** Each cell in a 4×4 grid contains a light switch which is initially off. Jonathan can toggle all the switches along a diagonal of the grid. Find the number of possible on/off configurations he can achieve. (The diagonal does not have to be a main diagonal.)

Proposed by: Muztaba Syed

Solution. $\boxed{4096}$

If we checkerboard the grid, cells with different colors don't interact. So just focus on white cells. The corners can be anything, since we can toggle them individually. Inspecting the other 6 cells, we see they form a 2×3 grid. If the two rows are toggled to the same state, there are 2^3 ways. If they are toggled to different states, there are also 2^3 ways. In total there are $2 \cdot 2 \cdot (2^3 + 2^3) = 2^6$ ways for the white cells. The answer is then $(2^6)^2 = \boxed{4096}$. □

11. **[12]** Let $ADMITS$ be a cyclic equiangular hexagon. Let U be the intersection of segments $\overline{MS}$ and $\overline{AI}$. Given that $\overline{IS} = 14$ and $\overline{ST} = 10$, find the sum of areas $[SAM] + [TSUI]$.

Proposed by: James Wu

Solution. $\boxed{54\sqrt{3}}$

First, since $\angle STI$ is 120 degrees, by Law of Cosines we have $\overline{IT} = 6$.

Let's consider the quadrilateral $SADM$. Since $\overline{AD}$ is shared, and the quadrilateral is cyclic which makes $\angle ASD = \angle AMD$, combined with the equiangular condition we have

$$\triangle SAD \cong \triangle MDA$$

Similarly, we have

$$\triangle SAD \cong \triangle MDA \cong \triangle DMI \cong \triangle TIM \cong \triangle ITS \cong \triangle AST$$

This congruence gives

$$\overline{AS} = \overline{TI} = \overline{MD} = 6, \overline{ST} = \overline{IM} = \overline{DA} = 10$$

Since the triangles have the same height, this also makes $\overline{MS}$ parallel to $\overline{IT}$ and $\overline{AD}$, and similarly $\overline{AI} \parallel \overline{MD} \parallel \overline{ST}$

This makes $TSUI$ a parallelogram. Since $\angle TSU = 180° - 120° = 60°$, we have $\angle ASU = 120° - 60° = 60°$. This means that $\triangle SAU$ is an equilateral triangle. Similarly, we can prove that $\triangle IMU$ is an equilateral triangle.

Therefore, by Law of Sines we have

$$[SAM] = \frac{1}{2} \cdot 6 \cdot (6+10) \cdot \frac{\sqrt{3}}{2} = 24\sqrt{3}$$

And since the distance from S to $\overline{AI}$ is $6 \cdot \frac{\sqrt{3}}{2} = 3\sqrt{3}$, we also have

$$[TSUI] = 10 \cdot 3\sqrt{3} = 30\sqrt{3}$$

Which makes our answer

$$24\sqrt{3} + 30\sqrt{3} = \boxed{54\sqrt{3}}.$$

□

12. **[12]** Let a, b, c, and d be integers satisfying the equations

$$a^2 + bd + c = 0 \qquad b^2 + ca + d = 1 \qquad c^2 + db + a = 2 \qquad d^2 + ac + b = 3$$

Find all possible ordered tuples (a, b, c, d).

Proposed by: Peter Bai

Solution. $\boxed{(1, 0, -1, 2)}$

Notice that the bd term appears in both of the first and third equations. We can subtract the first equation from the third to get

$$c^2 - a^2 + a - c = 2 \implies (c-a)(c+a) + (a-c) = 2 \implies (c+a-1)(c-a) = 2$$

We can now use SFFT. Since both of a and c are integers, both factors on the LHS of the last equation must be integers as well. This means that we must have

$$(c+a-1, c-a) \in \{(2,1),(1,2),(-2,-1),(-1,-2)\}$$

Solving for a and c in each of the 4 possibilities gives us our possible solutions as

$$(a,c) \in \{(1,2),(0,2),(0,-1),(1,-1)\}$$

Since the given equations are symmetric with respect to the pairs of variables (a,c) and (b,d), and since subtracting the second equation from the third gives an RHS of $4-2=2$, analogous logic gives us

$$(b,d) \in \{(1,2),(0,2),(0,-1),(1,-1)\}$$

as well.

Now, we need to see which combinations of (a,c) and (b,d) work. Notice that, by taking the difference of two equations as our first step, we have guaranteed that c^2+a and a^2+c differ by exactly 2. The original equations also require $a^2+c=-bd$ and $c^2+a=2-bd$, but we have now shown that the second one is implied by the first one, so we only need that $a^2+c=-bd$. Similarly, $b^2+d=1-ac$ must be satisfied as well.

For each possible (a,c), a table of the corresponding value of a^2+c and $-ac$ are provided below:

(a,c)	a^2+c	$-ac$
$(1,2)$	3	-2
$(0,2)$	2	0
$(0,-1)$	-1	0
$(1,-1)$	0	1

The only (a,c) with a a^2+c that has a possible matching value for $-bd$ is $(1,-1)$, which gives us $-ac=1$ and $a^2+c=-bd=0$. This means that (b,d) is either $(0,2)$ or $(0,-1)$. Since $b^2+d=1-ac=2$, we must have $(b,d)=(0,2)$ and our only possible solution is $(a,b,c,d)=\boxed{(1,0,-1,2)}$. □

13. **[13]** Let p,q be distinct primes such that

$$k=\frac{p^2+15\cdot q^3}{p+15\cdot q}$$

is an integer. Find the sum of all possible k.

Proposed by: Ryan Tang

Solution. $\boxed{47}$

We first notice that $(q,p+15\cdot q)=(p,q)=1$. We also have $p^2+15q^3 \pmod{p+15q} \equiv 225q^2+15q^3 \equiv 15q^2(q+15)$, so $15(q+15)\equiv 0 \pmod{p+15\cdot q}$. Now, suppose $(p,15)\neq 1$, then we have $p+15q \mid q+15$. Due to size restrictions, this is impossible. If $p=3$, then $1+5q \mid 5(q+15)$. Then, $1+5q \mid 74$. This is impossible. If $p=5$, then $1+3q \mid 3(q+15)$. Then, $1+3q \mid 44$, so we clearly have $q=7$. By inspection, this works. Hence, $k=\frac{5^2+15\cdot 7^3}{5+15\cdot 7}=47$. Since this is the only possible value, $\boxed{47}$ is the answer. □

14. **[13]** An 8×8 checkerboard is colored black and white such that the top left cell is white. A set of 4 distinct cells of the checkerboard is called *stable* if

- the four cells are not all the same color,
- the centers of the four cells form a square with sides parallel to the gridlines, and
- the top right cell of the four is black.

Find the number of ways to select 16 stable sets such that no black cell is in two sets.

Proposed by: Muztaba Syed

Solution. $\boxed{3456}$

If the top right cell is black, then so is the bottom left cell. This means the top left and bottom right cells are both white. A square is uniquely determined by the two black cells, which must be on the same diagonal.

The diagonals of black cells have lengths 2, 4, 6, 8, 6, 4, and 2. In the diagonals with length 2, there is 1 way to pair the black cells into endpoints of a square. In the diagonal of length 4 there are 2 ways, since we need to make sure the square has white vertices (so the two black cells we pick must be an even distance apart). The diagonal of length 6 has $3 \cdot 2 = 6$ ways, and the diagonal of length 8 has $4 \cdot 3 \cdot 2 = 24$ ways. In total the answer is

$$1 \cdot 2 \cdot 6 \cdot 24 \cdot 6 \cdot 2 = \boxed{3456}.$$

□

15. **[14]** Let $ABCDE$ be a convex pentagon such that $DE = EA = AB = BC$ and $DE \parallel CB$. Suppose $\angle EAB = 120^\circ$, $CD = 11$, and triangle ACD has area 27. Find $AD^2 + AC^2$.

Proposed by: Muztaba Syed

Solution. $\boxed{121 + 36\sqrt{3}}$

Construct parallelogram $DEAX$. Then note that $ABCX$ is also a parallelogram, so $XD = XC = XA$. Thus A is the circumcenter of ACD. Additionally $\angle DXC = 120^\circ$, so $\angle DAC = 60^\circ$.

Let the lengths AD and AC be a and b. Then by sine area formula and Law of Cosines we see $a^2 + b^2 - ab = 121$ and $ab = 27 \cdot \frac{4}{\sqrt{3}} = 36\sqrt{3}$. So $a^2 + b^2 = \boxed{121 + 36\sqrt{3}}$. □

16. **[TIEBREAKER]** In a 5×5 grid, each cell is colored either white or black. No 2×2 square contains all cells of the same color. Furthermore, all white cells are connected, i.e. one can walk from any white cell to any other white cell by only walking through edge-adjacent white cells. The same is true of black cells. Estimate the number of such grids, where rotations and reflections are considered distinct. If your answer is A and the true answer is E, your score on this tiebreaker will be $\min\left(\frac{E}{A}, \left(\frac{A}{E}\right)^{1.5}\right)$. A higher tiebreaker score is better.

Proposed by: Jerry Xu

Solution. $\boxed{660}$

□