Team Round Solutions

LMT Fall 2025

December 13th, 2025

1. **[20]** Let $ABCD$ be a square with side length 1. Circle ω is tangent to AB and AD, and passes through C. Find the area of ω.

   Proposed by: Jerry

   Solution. $\boxed{(6-4\sqrt{2})\pi}$

   Let r be the radius of the circle. Then $r + r\sqrt{2} = \sqrt{2}$, so the area is

$$\left(\frac{\sqrt{2}}{1+\sqrt{2}}\right)^2 \pi = \left(\sqrt{2}(\sqrt{2}-1)\right)^2 \pi = (6-4\sqrt{2})\pi.$$

□

2. **[20]** Let x and y be positive integers satisfying $x^y = 2^{18}$. Find the smallest possible value of $x+y$.

   Proposed by: Jonathan

   Solution. $\boxed{13}$

   The possibilities for (x, y) are:

$$\begin{aligned}(2,18) &\implies x+y=21\\ (4,9) &\implies x+y=13\\ (8,6) &\implies x+y=14.\end{aligned}$$

   We could also have $x = 2^6, 2^9$, or 2^{18}, but the sum $x+y$ would clearly get quite cooked. The smallest value we see is $\boxed{13}$, so that is our answer. □

3. **[20]** Find the number of ways to color a 3×3 grid with 3 colors such that each color is used exactly 3 times and no cells of the same color share an edge.

   Proposed by: Samuel Tsui

   Solution. $\boxed{36}$

   Consider the center square which can be colored in 3 ways. Then we have 6 ways to color the remaining two cells of that color. Finally picking another cell and coloring it in one of 2 ways fixes the rest of the grid. Thus the answer is $3\cdot6\cdot2=\boxed{36}$. □

4. **[20]** Ben and Jerry are swimming across a really big pool. It takes 40 minutes for Ben to travel the length of the pool and 30 minutes for Jerry to travel the length of the pool. As soon as either of them reach one end of the pool, they turn around and start swimming back towards the other end. If Ben and Jerry simultaneously start swimming in the same direction from one end of the pool, find the number of minutes it takes before they meet again.

   Proposed by: Jerry Xu

Solution. $\boxed{\dfrac{240}{7}}$

Label one end of the pool (their starting point) as A, and the other end of the pool as B. The first time they meet after the start will happen when Jerry has already reached B and turned around back towards A, but before Ben does so. When Jerry reaches B, 30 minutes have elapsed, and Ben is $\frac{3}{4}$th of the way from A to B. Then, Ben and Jerry are swimming towards each other, and need to cover $\frac{1}{4}$th of the length of the pool. The number of minutes this takes is

$$\frac{\frac{L}{4}}{\frac{L}{30}+\frac{L}{40}} = \frac{30}{7}.$$

Our answer is $30+\frac{30}{7} = \boxed{\dfrac{240}{7}}$. □

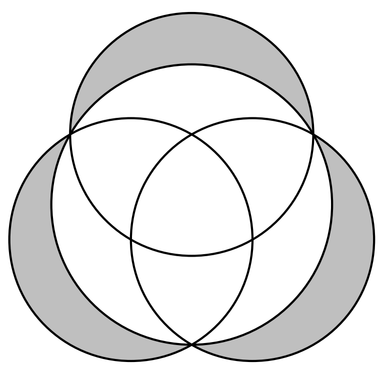
5. **[30]** Let f be a function satisfying $f(x)+f\left(1-\frac{1}{x}\right) = x$. Find $f(2)$.

Proposed by: Samuel Tsui

Solution. $\boxed{\dfrac{1}{4}}$

Plugging in $x = 2, \frac{1}{2}, -1$ we have $f(2)+f\left(\frac{1}{2}\right) = 2$, $f\left(\frac{1}{2}\right)+f(-1) = \frac{1}{2}$, $f(-1)+f(2) = -1$. Thus $f(2) = \frac{2-1-\frac{1}{2}}{2} = \boxed{\dfrac{1}{4}}$ □

6. **[35]** Three circles with radius 2 are drawn such that their centers form an equilateral triangle with side length 2. These circles intersect with each other at exactly 6 points, 3 of which are the centers of the circles. Another circle is drawn passing through the other 3 intersections. Find the area of the shaded region.



Proposed by: Jonathan

Solution. $\boxed{\dfrac{2\pi}{3}+4\sqrt{3}}$

We can consider the shaded area has the sum of 3 semicircles with radius 2 plus an equilateral triangle with side length 4 minus a big circle. The total area of these three semicircles is 6π, the equilateral triangle $4\sqrt{3}$, and big circle $\frac{16}{3}\pi$ Thus, our shaded area is $6\pi+4\sqrt{3}-\frac{16}{3}\pi = \boxed{\dfrac{2\pi}{3}+4\sqrt{3}}$. Note that we cannot find the area of the sector by considering a 60° sector of a circle with radius 4, which would give us an incorrect answer of $12\sqrt{3}-6\pi$. □

7. **[40]** Eddie is at $(0,0)$ on the coordinate plane and wants to get to $(2,10)$. If he is at (x,y), he can move to one of $(x+2,y)$, $(x-2,y)$, $(x+1,y+1)$, $(x-1,y+1)$. He does not revisit any points along his path and his x and y coordinates are always greater than or equal to 0 and less than or equal to 2 and 10 respectively. Find the number of ways Eddie can get to $(2,10)$.

Proposed by: Samuel Tsui

Solution. $\boxed{1024}$

Observe his path must go through $(1,1),(1,3),(1,5),(1,7),(1,9)$. Note there are 2 ways to get from $(0,0)$ to $(1,1)$ and $(1,9)$ to $(2,10)$ and there are 4 ways to get from $(1,i)$ to $(1,i+2)$ for $i=1,3,5,7$. Thus the answer is $2^2 \cdot 4^4 = \boxed{1024}$. □

8. **[40]** ST, GT, and CT are 3 friends that are each given 3 unique positive integers from 1-9 inclusive such that none of them share any numbers. They then have the following perfectly logical conversation.

- ST: My 3 numbers form a geometric sequence.
- GT: ST has the largest number, and I don't know if CT's numbers form an arithmetic sequence.

If GT's numbers are a,b,c with $a<b<c$, find $100a+10b+c$.

Proposed by: Alexander Duncan

Solution. $\boxed{257}$

If ST's numbers form a geometric sequence, then he must have $(1,2,4)$, $(2,4,8)$, $(1,3,9)$, or $(4,6,9)$. GT knows that ST has the largest number, so ST must have $(1,3,9)$ or $(4,6,9)$. Also, if GT knew exactly what ST had, then he would also know what CT has so he would know for certain whether CT's numbers form an arithmetic sequence. Thus, from GT's perspective, it must be possible for ST to have $(1,3,9)$ or $(4,6,9)$, so he can't have 1, 3, 4, or 6. He also can't have 9 because ST has 9. Additionally, GT must have 2 because otherwise it would be possible for ST to have $(1,2,4)$. Taking this all into consideration, GT must have 2 as well as two of the following three numbers: 5, 7, and 8. Additionally, CT must have the number GT is missing as well as either 1 and 3 or 4 and 6. If GT is missing 5, then he would know that CT's numbers form an arithmetic sequence. If GT is missing 7, then he would know that CT's numbers do not form an arithmetic sequence. If GT is missing 8, then if CT has 1 and 3 then his numbers wouldn't form an arithmetic sequence but if CT has 4 and 6 then his numbers would. Thus, GT must be missing 8 so he has $2,5,7$ for a final answer of $\boxed{257}$. □

9. **[45]** Given

$$(x-21)(x-22)(x-23)(x-24) = 2025,$$

find the sum of all possible values of $(x-20)(x-25)$.

Proposed by: James Wu

Solution. $\boxed{-10}$

Notice that we can pair $A=(x-21)(x-24)=x^2-45x+504$, $B=(x-22)(x-23)=x^2-45x+506$, and we want $C=(x-20)(x-25)=x^2-45x+500$. Since $A=C+4$ and $B=C+6$, we can rewrite the original equation as

$$(C+4)(C+6) = 2025 \Longrightarrow C^2+10C-2001 = 0$$

By Vieta's formula, the sum of all possible C is $\boxed{-10}$. □

10. **[45]** A circular chip of radius $\frac{1}{2}$ is placed uniformly at random on an infinite grid of unit squares. Find the expected number of unit squares that the chip at least partially covers.

Proposed by: Alexander Duncan

Solution. $\boxed{3+\dfrac{\pi}{4}}$

Note that it is only possible for the square which the center of the circle is contained in as well as the 8 other squares that share a vertex with that center square to be covered. The probability that a square that shares an edge with the center square is covered is $\frac{1}{2}$ as distance between the center of the circle and the shared edge must be less than $\frac{1}{2}$. Additionally, the probability that a square that shares a vertex but not a edge with the center square is covered is $\frac{\pi}{16}$ as it must be inside the quarter circle with center at the shared vertex and radius of $\frac{1}{2}$. Also, the center square must be covered, so the answer is $1+4\cdot\frac{1}{2}+4\cdot\frac{\pi}{16} = \boxed{3+\dfrac{\pi}{4}}$ □

11. **[50]** Let $ABCD$ be a parallelogram and ω be a circle with radius 2. Suppose ω is tangent to AB, AD, BC at X, Y, C respectively and intersect CD at Z. If $AX = 1$, find DZ.

Proposed by: Samuel Tsui

Solution. $\boxed{\frac{9}{5}}$

Note that CY is the diameter of ω so $CY = 4$. Then letting $BC = BX = x$ by Pythagorean Theorem we have $(x-1)^2 + 16 + (x+1)^2$ so $x = 4$. Thus by Power of a Point $DZ = \frac{DY^2}{CD} = \boxed{\frac{9}{5}}$. □

12. **[50]** Let f be a function such that $f(0) = 1$ and for positive integers x,

$$f(x) = \begin{cases} 0 & \text{if } x \equiv 0 \pmod 3 \\ \frac{1}{3} f\left(\frac{x-1}{3}\right) & \text{if } x \equiv 1 \pmod 3 \\ \frac{1}{9} f\left(\frac{x-2}{3}\right) & \text{if } x \equiv 2 \pmod 3 \end{cases}.$$

Find $\sum_{n=1}^{\infty} f(n)$.

Proposed by: Samuel Tsui

Solution. $\boxed{\frac{4}{5}}$

Consider n in base 3. Then $f(n)$ is 0 if n contains a 0 and is $\frac{1}{3^{\text{sum of digits}}}$ if n only contains 1 and 2. Thus considering the contribution of each digit $\sum_{n=1}^{\infty} f(n) = \sum_{k=1}^{\infty} \left(\frac{4}{9}\right)^k = \boxed{\frac{4}{5}}$. □

13. **[55]** Let ℓ be a line. Pick points A,B on the same side of ℓ such that the distances to ℓ are 7,3 and $AB = 5$. Distinct points S, T are chosen on ℓ such that (ASB), (ATB) are tangent to ℓ. Find ST.

Proposed by: Ryan Tang

Solution. $\boxed{\frac{5\sqrt{21}}{2}}$

Let M be the intersection of AB and ℓ. By power of a point on M, we have $MS^2 = MA \cdot MB = MT^2$. Furthermore, by similar triangles, $AM = \frac{7 \cdot 5}{4}$ while $MB = \frac{3 \cdot 5}{4}$. Hence, the answer is $2 \cdot \sqrt{AM \cdot MB} = \frac{5\sqrt{21}}{2}$. □

14. **[60]** Let a_n be a sequence such that $a_1 = 1$ and

$$a_n = 1 + \frac{2025}{a_{n-1}} + \frac{2025}{a_{n-1}a_{n-2}} + \cdots + \frac{2025}{a_{n-1}a_{n-2}\cdots a_1}$$

Find $\lfloor a_{2025} \rfloor$.

Proposed by: Samuel Tsui

Solution. $\boxed{45}$

Multiplying both sides by $a_{n-1}a_{n-2}\cdots a_1$ gives $a_n a_{n-1}\cdots a_1 = a_{n-1}a_{n-2}\cdots a_1 + 2025(a_{n-2}a_{n-3}\cdots a_1 + a_{n-3}a_{n-4}\cdots a_1 + \cdots + a_1 + 1)$. Letting $b_n = a_n \ldots a_1$ we have $b_n = b_{n-1} + 2025(b_{n-2} + b_{n-3} + \cdots + b_1 + 1) = b_{n-1} + 2025b_{n-2} + (b_{n-1} - b_{n-2}) = 2b_{n-1} + 2024b_{n-2}$. Since we know $b_1 = 1$ and $b_0 = 1$, we can solve this recurrence yielding $b_n = \frac{46^n + (-44)^n}{2}$. Thus $\lfloor a_{2025} \rfloor = \left\lfloor \frac{b_{2025}}{b_{2024}} \right\rfloor = \left\lfloor \frac{46^{2025} + (-44)^{2025}}{46^{2024} + (-44)^{2024}} \right\rfloor = \boxed{45}$. □

15. **[70]** A bag has 5 red marbles, 3 green marbles, and 2 blue marbles. Marbles are drawn from the bag one at a time, with replacement. Find the expected number of draws that are required for each color to appear at least once, given that red appears first, then green, then blue.

Proposed by: Jerry Xu

Solution. $\boxed{8}$

Let $\mathbb{E}[T_i]$ denote the expected number of draws required for a new color to be drawn from the bag, given that $i-1$ colors have already been drawn. We desire

$$\mathbb{E}[T_1+T_2+T_3 \mid R \to G \to B] = \sum_{i=1}^{3} \mathbb{E}[T_i \mid R \to G \to B].$$

We know that

$$\mathbb{E}[T_1 \mid R \to G \to B] = 1$$

since the first marble must be red, which is the first time that color has been drawn. Then,

$$\mathbb{E}[T_2 \mid R \to G \to B] = \frac{1}{1-\frac{1}{2}} = 2$$

since we know the first non-red marble must be green, so we need the expected number of draws until a non-red appears. Finally,

$$\mathbb{E}[T_3 \mid R \to G \to B] = \frac{1}{\frac{2}{10}} = 5$$

because the last marble must be blue, so we count the expected number of draws until a blue appears. Our answer is

$$1+2+5 = \boxed{8}.$$

□