Speed Round Solutions

LMT Fall 2025

December 13th, 2025

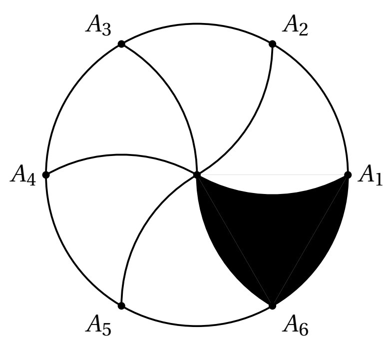
1. **[6]** This year, 2025, is a special year because $2025 = 45^2$ is a perfect square. Find the next year after 2025 that is also a perfect square.

   Proposed by: Peter Bai

   Solution. $\boxed{2116}$

   We compute $46^2 = \boxed{2116}$. □

2. **[6]** The points A_1, A_2, A_3, A_4, A_5, and A_6 are equally spaced on the circumference of a unit circle. Find the area of the shaded region in the diagram.



   Proposed by: William Hua

   Solution. $\boxed{\frac{\pi}{6}}$

   □

3. **[6]** Alice and Bob, two LHS students, are trying to walk from Commons II to the Math building. Alice walks 4 meters north, then 9 meters west, then 8 meters north, at a speed of 1 meter per second. Bob departs at the same time as Alice, but walks in a straight line to the Math building at 0.5 meters per second. Find the absolute difference in their arrival times, in seconds.

   Proposed by: Lena Lee

   Solution. $\boxed{27}$

   The student traveling along the dirt path will travel 15m in total, since $\sqrt{(8+4)^2+(9)^2} = 15$: therefore, this student will take 15 seconds to get across. The student walking along the hallways will travel $4+9+8 = 21$ meters in total, and will therefore take 42 seconds to travel that distance. Thus, the answer is $42-15 = 27$ seconds faster. □

4. **[6]** Jerry rolls two six-sided dice. The probability that the product of the two numbers that he rolled is even is p, and the probability that the sum of the two numbers that he rolled is even is q. Find $\frac{p}{q}$.

   Proposed by: Atticus Oliver

Solution. $\boxed{\frac{3}{2}}$

Regardless of what Jerry rolls on the first die, the second die has a $\frac{1}{2}$ chance of rolling a number that would make the sum even. For the product, both numbers must be odd to get an odd product, so the probability of an odd product is $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$, so the probability that it's even is $\frac{3}{4}$. Therefore the ratio of the probabilities is $\frac{\frac{3}{4}}{\frac{1}{2}} = \boxed{\frac{3}{2}}$. □

5. **[6]** Jame S. the window-washer is washing the windows of the new Lexington High School building. He works from 6AM to 10AM and he washes the windows from the 6th to the 10th floor. Find the average number of minutes it takes for him to wash one floor's windows.

   Proposed by: Rohan Danda

   Solution. $\boxed{48}$

   Jame S. cleans 5 floors every 4 hours. This means for one floor, it will take $\frac{4}{5}$ of an hour, or $\boxed{48}$ minutes. □

6. **[6]** Let $\{a_n\}$ be a sequence such that $a_1 = 15, a_2 = 18$, and $a_n = a_{n-1} - a_{n-2}$ for all $n \geq 3$. Find the sum of all distinct terms in the sequence.

   Proposed by: Danyang Xu

   Solution. $\boxed{0}$

   We have $a_1 = 15$, $a_2 = 18$, $a_3 = 3$, $a_4 = -15$, $a_5 = -18$, $a_6 = -3$, and $a_{i+6} = a_i$. Thus the answer is $15+18+3-15-3-18 = \boxed{0}$. □

7. **[6]** Find the sum of all 2-digit numbers for which the sum of the tens digit, the ones digit, and the product of its 2 digits all sum to itself. For example, 19 satisfies these conditions because $1+9+1\cdot 9 = 19$.

   Proposed by: Vedant Joshi

   Solution. $\boxed{531}$

   Let a and b be the tens and ones digit respectively. We create the equation $a+b+ab = 10a+b$, so $ab = 9a$. Since a must be non-zero, we can further simplify to get $b = 9$. Thus, we want to find the sum of all 2-digit numbers ending in 9. This is $9\cdot 9 + 10\cdot(1+\cdots+9) = 531$. □

8. **[6]** Define $a \star b$ to be $ab + a - b$. Find

   $$(((2\star 3)\star 4)\star 5)\star 6.$$

   Proposed by: James Wu

   Solution. $\boxed{841}$

   Notice that

   $$a\star b = ab + a - b = (a-1)(b+1)+1$$

   so $2\star 3\star 4\star 5\star 6$ is

   $$(2-1)(3+1)(4+1)(5+1)(6+1)+1 = \boxed{841}$$

   □

9. **[6]** Let Ω be a circle with center O and radius 1. A semicircle ω with diameter AB is constructed such that A and B lie on circle O, O lies on ω, and O does not lie on line AB. Find the area of the region contained by the minor arc AB on Ω and arc AOB on ω.

   Proposed by: Alexander Duncan

Solution. $\boxed{\dfrac{\pi-1}{2}}$

We can rewrite the area of this region as area of the minor sector AB in circle O minus the area of triangle AOB plus the area of the semicircle. Note that $\angle AOB = 90^\circ$ so $AB = \sqrt{2}$. The area of the sector is $\frac{\pi}{4}$, the area of the triangle is $\frac{1}{2}$, and the area of the semicircle is $\frac{\pi}{4}$ so the area of this region is $\boxed{\dfrac{\pi-1}{2}}$ □

10. **[6]** 6 distinct books are distributed among 3 students A, B, and C. Find the number of ways the books can be distributed such that one student gets 4 books, while the remaining students get 1 each.

Proposed by: Jonathan Liu

Solution. $\boxed{90}$

There are $\binom{6}{4}\binom{2}{1}\binom{1}{1} = 30$ ways to distribute the books in a $(4,1,1)$ distribution to some distinct students A, B, C. We then multiply by 3 since to consider which person gets 4 books, for our answer of $3 \cdot 30 = \boxed{90}$. □

11. **[6]** Arnab performs the following addition in base b:

$$\begin{array}{cccc} & S & S & 8_b \\ + & T & 5 & T_b \\ \hline 1 & 0 & 0 & 0_b \end{array}$$

where S, T, 5, and 8 are digits in base b. Find $S+T+b$ in base 10.

Proposed by: Samuel Tsui and Arnab Dasgupta

Solution. $\boxed{25}$

We have $T+8=b$, $S+6=b$, $S+T=b-1$ and solving gives $S=7$, $T=5$, $b=13$ for an answer of $7+5+13=\boxed{25}$. □

12. **[6]** The point $(3,4)$ is rotated around the point $(0,0)$ by some angle to point A. Point A is then rotated around the point $(3,0)$ by some angle to point B. Find the area of the region bounded by all possible B.

Proposed by: Alexander Duncan

Solution. $\boxed{60\pi}$

The distance between A and $(3,0)$ can be anywhere from 2 to 8, so the area is $64\pi - 4\pi = \boxed{60\pi}$ □

13. **[6]** In a group of 6 people, each pair of people is either friends or enemies. Find the number of ways for every person to have exactly 4 friends.

Proposed by: Alexander Duncan

Solution. $\boxed{15}$

There must be exactly 3 pairs of people who are enemies, so the number of ways is $\frac{\binom{6}{2}\cdot\binom{4}{2}\cdot\binom{2}{2}}{3!} = \boxed{15}$. □

14. **[6]** Jerry chooses a random line that passes through the origin in the Cartesian plane. Find the probability that this line passes through the square with vertices at the points $(1,1,1)$, $(1,-1,1)$, $(-1,-1,1)$, and $(-1,1,1)$.

Proposed by: Atticus Oliver

Solution. $\boxed{\dfrac{1}{3}}$

Notice that this is one of the faces of the cube centered at the origin with side length 2. Also notice that any line that passes through the center of this cube will pass through two opposite faces. There are three rotationally symmetrical pairs of opposite faces, so the probability that it passes through any given pair (and therefore any given face) is just $\boxed{\dfrac{1}{3}}$. □

15. **[6]** On the algebra tryout, any real score between $[0, 100]$ can be achieved. The average was 91, and the median was 100. Find the largest possible integer minimum of the grades.

Proposed by: Ryan Tang

Solution. $\boxed{81}$

Suppose there were n people and k was the lowest grade. The data set $100, 100, \cdots, 100, 100, k, \cdots, k$ would maximize k, where there are $n+r$ 100's and n k's where $r \in \{1, 2\}$. This gives us $k = 82 - \frac{9}{n} \cdot r$ so $k < 82$. Thus, picking $n = 9$ gives us 81 is achievable. □

16. **[6]** Find the number of subrectangles in a 6×6 grid that do not contain any cells in the middle 2×2.

Proposed by: Samuel Tsui

Solution. $\boxed{216}$

There are $\binom{3}{2} \cdot \binom{7}{2} = 63$ rectangles in each of the 4 border 2×6 with $\binom{3}{2} \cdot \binom{3}{2} = 9$ rectangles counted twice in each of the 4 corner 2×2. Thus the answer is $4 \cdot 63 - 4 \cdot 9 = \boxed{216}$. □

17. **[6]** In isosceles trapezoid $ABCD$ with $AD \parallel BC$, the reflection of A across BD lies on line BC. Given that $AD = 5, BC = 7$, find the area of trapezoid $ABCD$.

Proposed by: Ryan Tang

Solution. $\boxed{12\sqrt{6}}$

Suppose A' is the reflection. Then, since $\angle DA'C = 180 - \angle DAB = \angle BCD$, we have $A'DC$ is an isosceles triangle. Thus, since $AB = 5$, we have $CD = 5$. Thus, the height is $2\sqrt{6}$ and we are done. □

18. **[6]** A candy is placed in one cell of a 4×4 square grid. Sam does not know where the candy is. In one move, Sam can check if the candy is within a selected 2×2 square grid. Find the least number of moves Sam must make to ensure he knows exactly which cell the candy is in.

Proposed by: Ryan Tang

Solution. $\boxed{5}$

Check the L-shape first, which gives 3 checks. Then, we want to find the candy within a 2×2. Without loss of generality, suppose that this 2×2 is in the top left corner. Then, consider checking the square with corners $(0, 1), (2, 3)$ and the square with corners $(1, 2), (3, 4)$. It can be seen that this finishes.

To see why 4 can not be achieved: The result after the baby reaches into the cabinet each time is yes or no, giving $2^4 = 16$ possible such sequences. There is also $4 \times 4 = 16$ cells, which means that there must be a bijection between the cells and sequences. This means that at least one cell will be the result of YYYY, but then no matter where that cell is there must be at least 7 cells with the result of NNNN which can't be distinguished. □

19. **[6]** Find the number of up-right paths that start from the point $(0, 0)$ and end when first reaching the line $y = 5$ such that the path can't move in the same direction for more than 2 turns in a row.

Proposed by: Alexander Duncan

Solution. $\boxed{180}$

Let a_n be the number of ways to reach the line $y = n$ for the first time. $a_1 = 3$ and $a_2 = 9$, and for $n > 2$, $a_n = 2a_{n-1} + 2a_{n-2}$ as every time you reach the line $y = n$ for the first time, it must either be by going right 1 or 2 times on the line $y = n-2$ and then going up twice or by going right 1 or 2 times on the line $y = n-1$ and then going up once. Thus, we get $a_3 = 24$, $a_4 = 66$, and $a_5 = \boxed{180}$. □

20. **[6]** There exists a unique three-digit positive integer n that can be expressed in both $\overline{xxy}_{10}$ and $\overline{yxx}_b$, where $b < 10$. Find n, in base 10.

Proposed by: James Wu

Solution. $\boxed{445}$

First, we must have $5 \le b \le 9$ because if $b \le 4$, n wouldn't be a three-digit integer. By the definition of n we have

$$110x + y = yb^2 + xb + x \Longrightarrow x(109 - b) = y(b+1)(b-1)$$

Since $b - 1 < b + 1 \le 10$ and $y < b \le 9$, we can first check whether $109 - b$ has a prime factor exceeding 10. A quick check shows that $b = 9$ is the only base that works, giving

$$x \cdot 100 = y \cdot 10 \cdot 8$$

And thus $x = 4$, $y = 5$, so $n = \boxed{445}$. □

21. **[6]** Let ABC be a triangle such a that $AB = 3$, $BC = 4$, $AC = 5$. Let P be a point inside ABC such that $[PAB] + [PBC] = 2$. Find the minimum possible value of BP.

Proposed by: Samuel Tsui and Ryan Tang

Solution. $\boxed{\frac{4}{5}}$

Observe that $[PAC] = [ABC] - [PAB] - [PBC] = 6 - 2 = 4$. Since $AC = 5$, we have $\text{dist}(P, AC) = 2 \cdot \frac{[PAC]}{AC} = \frac{8}{5}$. In particular, the locus of P is a line contained in the triangle parallel to AC with distance $\frac{8}{5}$ to AC. Hence, $BP \ge \frac{4}{5}$. □

22. **[6]** For $n \ge 2$, define $f(n)$ as the remainder of $1 + 2 + \cdots + 2n$ when divided by $1 + 2 + \cdots + n$. Find $f(2) + f(3) + \cdots + f(14)$.

Proposed by: Ryan Tang

Solution. $\boxed{\binom{15}{3} = 455}$

We have that $\left\lfloor \frac{2n(2n+1)}{n(n+1)} \right\rfloor = \left\lfloor 4 - \frac{2}{n+1} \right\rfloor = 3$ for n, so $f(n) = n(2n+1) - 3 \cdot \frac{n(n+1)}{2} = \frac{n(n-1)}{2} = \binom{n}{2}$. Hence, $\sum_{k=2}^{14} \binom{n}{2} = \binom{15}{3}$ by hockeystick. □

23. **[6]** Find the triple of positive reals (x, y, z) that satisfies:

$$xy = x + 2y$$
$$yz = y + 3z$$
$$zx = z + 6x.$$

Proposed by: Ryan Tang

Solution. $\boxed{\left(\frac{37}{13}, \frac{37}{11}, \frac{37}{4}\right)}$

Divide by the LHS on each equation, and we get

$$1 = \frac{1}{y} + \frac{2}{x}$$
$$1 = \frac{1}{z} + \frac{3}{y}$$
$$1 = \frac{1}{x} + \frac{6}{z}$$

Solving them gives $\frac{1}{x} = \frac{13}{37}, \frac{1}{y} = \frac{11}{37}, \frac{1}{z} = \frac{4}{37}$. □

24. **[6]** Let ABC be a triangle with points P and Q on AB and AC respectively such that $BP = 6$, $AP = 7$, $CQ = 7$, $AQ = 5$. Let M be the midpoint of BC. If there exists a circle ω such that A, P, M, and Q all lie on the circumference of ω, find BC.

Proposed by: Samuel Tsui

Solution. $\boxed{18}$

Let $(APMQ)$ intersect BC again at X. By POP, we have the following two equations:

$$BX \cdot BM = BP \cdot BA = 78$$
$$CX \cdot CM = CQ \cdot CA = 84.$$

Adding the two, we get $BM = 9$ so $BC = 18$. □

25. **[6]** Let a partition of a sequence be beautiful if the sequence is partitioned into contiguous blocks that are cyclic rotations of $[1,2,\cdots,k]$, where k is a positive integer. For example, $[2,3,1,1,2]$ is beautiful as it can be partitioned into $[2,3,1]$ and $[1,2]$. Find the sum of the number of beautiful partitions over all sequences of length 9.

Proposed by: James Wu

Solution. $\boxed{2584}$

Let $f(n)$ be the sum of the number of beautiful partitions over all sequences of length n. We have

$$f(n) = \sum_{i=0}^{n-1} f(i) \cdot (n-i)$$

Where $f(0) = 1$. Define F_n as the Fibonacci sequence, where $F_0 = 0$ and $F_1 = 1$. We claim that for all positive integers n, we have

$$f(n) = F_{2n}$$

We can prove this by induction. First, we have $f(1) = 1 = F_2$. Assuming that for some positive integer k we have $f(a) = F_{2a}$ for all positive integers $a \le k$, we can rewrite the summation as

$$f(k+1) = \sum_{i=0}^{k} f(i) \cdot (k+1-i) = \sum_{i=0}^{k} f(i) + \sum_{i=0}^{k-1} f(i) \cdot (k-i)$$

The second sum is F_{2k}, and the first is F_{2k+1}. Therefore, we have $f(k+1) = F_{2k+1} + F_{2k} = F_{2k+2}$. So the answer is $F_{18} = \boxed{2584}$. □