LMT Fall 2025 Guts Round Solutions- Part 1

Team Name:

1. **[14]** Find the smallest prime number greater than 200.

Proposed by: Benjamin Yin

Solution. $\boxed{211}$

We check that $201 = 3 \cdot 67$, and $203 = 7 \cdot 29$, and $205 = 5 \cdot 41$, and $207 = 9 \cdot 23$, and $209 = 11 \cdot 19$. We also notice that 211 is prime. □

2. **[14]** Jerry selects 4 points on a plane. Find the maximum possible number of distinct triangles with vertices selected from Jerry's points.

Proposed by: Ryan Tang

Solution. $\boxed{4}$

By drawing a quick sketch, it is apparent that it is possible to select the 4 points so that every possible selection of 3 distinct points yields a valid triangle. The answer is then $\binom{4}{3} = \binom{4}{1} = \boxed{4}$. □

3. **[14]** Let $\overset{\frown}{AB}$ be a minor arc on the circle with measure $104°$. Let X be a point on the circle distinct from A and B. Find the positive difference between the maximum and minimum nonzero possible measures of $\angle AXB$.

Proposed by: William

Solution. $\boxed{76}$

If X is on the arc, then $\angle AXB = \frac{104}{2}^\circ = 52°$. Otherwise, it is $180° - 52° = 128°$. So the answer is $76°$. □

LMT Fall 2025 Guts Round Solutions- Part 2

Team Name:

4. **[16]** Lonzo creates a three-digit passcode using the digits 1, 2, 3, and 4. Find the number of passcodes that have an even sum of digits, where digits are not necessarily distinct.

Proposed by: Benjamin Yin

Solution. $\boxed{32}$

There are a total of $4^3 = 64$ possible three-digit passcodes. Each passcode may be paired with a corresponding passcode, where each digit x in the passcode is replaced with the digit $5 - x$. Each pairing contains a passcode with an even and odd sum. Since we have 32 pairs, we have $\boxed{32}$ possible passcodes. □

5. **[16]** On a planet there are three groups of aliens: "Jerries", "Bens", and "Peters". One day, at a meeting, 4 Jerries, 8 Bens, and 10 Peters show up. Given that each alien shakes hands with every other alien not from their own group, find the number of total number of handshakes that occur.

Proposed by: Vedant Joshi

Solution. $\boxed{152}$

There will be 32 handshakes between Jerries and Bens, 80 handshakes between Bens and Peters, and 40 handshakes between Jerries and Peters. Thus, our answer is $32 + 80 + 40 = \boxed{152}$. □

6. **[16]** Lebron and Jordan both take x free throw attempts, with Lebron making 50% of them while Jordan makes 0%. Then, Lebron shoots $2y$ more attempts and makes 60% of them, while Jordan shoots $3y$ more attempts and also makes 60%. Given they end up making the same percentage of all of their shots, find the minimum possible value of $x+y$.

Proposed by: Samuel Tsui

Solution. $\boxed{11}$

We have $\frac{0.5x+1.2y}{x+2y} = \frac{1.8y}{x+3y}$ and solving yields $5x = 6y$. Thus the smallest solution would be $x = 6,\ y = 5$ which works so the answer is $6+5 = \boxed{11}$. □

LMT Fall 2025 Guts Round Solutions- Part 3

Team Name:

7. **[18]** Let $ABCD$ be a parallelogram and M the midpoint of CD. Given that $AM = 3, BM = 4, AB = 5$, find BC.

Proposed by: Ryan Tang

Solution. $\boxed{\frac{5}{2}}$

Consider a point M' on segment AB such that lines MM' and BC are parallel. Then, $\overline{BC} = \overline{MM'}$, and segment MM' is the median from M to segment AB. Since AMB is a 3-4-5 right triangle, $\overline{BC} = \overline{MM'} = \overline{M'B} = \frac{1}{2}\overline{AB} = \boxed{\frac{5}{2}}$. □

8. **[18]** There exist real numbers a and b such that $a+\frac{1}{b} = 7$ and $b+\frac{1}{a} = 6$. Find the sum of all possible values of ab.

Proposed by: Atticus Oliver and Danyang Xu

Solution. $\boxed{40}$

Multiplying the two equations together gives $ab+\frac{a}{a}+\frac{b}{b}+\frac{1}{ab} = 42 \Rightarrow ab+\frac{1}{ab} = 40$. Let $c = ab$. Then, we have that $c^2-40c+1 = 0$. Vieta's gives our answer of 40. □

9. **[18]** Today, 12/13/25, has a special property: The sum of the month (12) and the day (13) is equal to the last two digits of the year (25). Find the number of distinct dates in the 21st century with this property.

Proposed by: Vedant Joshi

Solution. $\boxed{365}$

Every date will work, as there exists a year whose last 2 digits are equal to the sum of the month number and the day number. However, note that February 29th will not work, as the sum would be $2+29 = 31$, but leap years only happen every 4 years on years which have a remainder of 0 when divided by 4. Thus, our answer is $366-1 = \boxed{365}$. □

LMT Fall 2025 Guts Round Solutions- Part 4

Team Name:

10. **[20]** Find the number of rearrangements of the word *MATHTEAM* that contain either the substring *MATH* or the substring *HEAT*.

Proposed by: Jonathan

Solution. $\boxed{178}$

We can treat subsets of words as blocks and use PIE. Words that contain *HEAT*: Use block $[HEAT]$ and leftovers $M, A, T, E \rightarrow 5! = 120$ ways. Words that contain *MATH*: Use block $[MATH]$ and leftovers $M, M, A, T \rightarrow \frac{5!}{2} = 60$ ways. Words that contain both: There is only 1 H, so the only way is to use block $[MATHEAT]$ and leftover $M \rightarrow 2! = 2$ ways. Thus, our answer is $60 + 120 - 2 = \boxed{178}$. □

__________ 11. **[20]** Find the largest prime factor of 142857.

Proposed by: Alexander Duncan

Solution. $\boxed{37}$

Note that $0.\overline{142857} = \frac{1}{7}$, so we have that $142857.\overline{142857} = \frac{1000000}{7}$ and $142857 = \frac{999999}{7}$. $999999 = 999 \cdot 1001 = (3^3 \cdot 37) \cdot (7 \cdot 11 \cdot 13)$ so $142857 = 3^3 \cdot 11 \cdot 13 \cdot 37$, giving an answer of $\boxed{37}$. □

__________ 12. **[20]** In triangle $ABC, \angle A = 72^\circ, \angle B = 68^\circ, \angle C = 40^\circ$. Let the incenter of $\triangle ABC$ be I and let $\overline{AI}$ intersect BC at point D. The tangent from D to the incircle of ABC intersects $\overline{AC}$ at point E. Find $\angle IEC$.

Proposed by: Benjamin Yin

Solution. $\boxed{146^\circ}$

The angles in $\triangle ABD$ sum to 180°, so $\angle ADB = 180^\circ - 68^\circ - \frac{72^\circ}{2} = 76^\circ$. Since $\overline{DB}$ and $\overline{DE}$ are both tangents of the incircle, $\angle ADE = \angle ADB = 76^\circ$. $\angle EDC = 180^\circ - 76^\circ - 76^\circ = 28^\circ$. From exterior angle theorem, $\angle DEA = 28^\circ + 40^\circ = 68^\circ$. Since $\overline{EA}$ and $\overline{ED}$ are both tangents, $\angle AEI = \angle DEI = \frac{68^\circ}{2} = 34^\circ$. Therefore, $\angle IEC = 180^\circ - \angle AEI = 180^\circ - 34^\circ = \boxed{146^\circ}$. □

13. **[22]** Let $TNAG$ be a square where $\overline{GT} = 6$. A point C is chosen so that $\overline{CT} \le 12$. Given that quadrilateral $CNAG$ is convex, find the area of the region bounded by all possible C.

Proposed by: James Wu

Solution. $\boxed{60\pi + 36\sqrt{3} + 18}$

We can use coordinates $T = (0,0), N = (0,6), A = (6,6), G = (6,0)$. By the definition of a convex polygon and the length of $\overline{CT}$, the following inequalities must be satisfied

$$\begin{cases} x \le 6 \\ y \le 6 \\ x + y \le 6 \\ x^2 + y^2 \le 144 \end{cases}$$

If you graph this out, the area formed can be split into a sector of a circle with center $(0,0)$, one isosceles right triangle, and two more right triangles. Let the intersection between $x = 6$ and $x^2 + y^2 = 144$ be P and the intersection between $y = 6$ and $x^2 + y^2 = 144$ be Q. We have $\overline{QN} = \overline{GP} = 6\sqrt{3}$ by the Pythagorean theorem, as well as $\angle QTN = \angle GTP = 60^\circ$. Then, the area of the sector is

$$12^2 \cdot \pi \cdot \frac{\angle QTP}{360^\circ} = 12^2 \cdot \pi \cdot \frac{150^\circ}{360^\circ} = 12^2 \cdot \pi \cdot \frac{5}{12} = 60\pi$$

. The area of $\triangle NTG$ is

$$\frac{6 \cdot 6}{2} = 18$$

And the area of $\triangle QTN = \triangle GTP$ is

$$\frac{6 \cdot 6\sqrt{3}}{2} = 18\sqrt{3}$$

Therefore, the total area is

$$60\pi + 18 + 18\sqrt{3} \cdot 2 = \boxed{60\pi + 36\sqrt{3} + 18}.$$

□

14. **[22]** Find the number of ways to color each cell of 4×4 grid red or blue such that there are 8 cells of each color and each row and column has at least 3 cells of the same color.

Proposed by: Samuel Tsui

Solution. $\boxed{144}$

There are $\binom{4}{2} = 6$ ways to choose which rows have 3 red cells then $4 \cdot 3 = 12$ ways to fill those rows and 2 ways to fill the last two rows. Thus the answer is $6 \cdot 12 \cdot 2 = \boxed{144}$. □

15. **[22]** Let a and b be integers with a greater than b satisfying $a^2b^2 + 2ab + 9 = a^2 + b^2$. Find all possible values of (a, b)

Proposed by: Samuel Tsui

Solution. $\boxed{(3,0), (4,-1), (1,-4)}$

Let $x = ab$ and $y = a - b$. Rearranging we get $y^2 - x^2 = 9$ so $(x, y) = (\pm 4, \pm 5), (0, \pm 3)$ giving solutions $\boxed{(3,0), (4,-1), (1,-4)}$. □

LMT Fall 2025 Guts Round Solutions- Part 6

Team Name:

______________ 16. **[24]** Real numbers a, b, c, d satisfy $a^2 + b^2 = 4$ and $(c-5)^2 + (d-12)^2 = 9$. Find the minimum value of $(c-a)^2 + (d-b)^2$.

Proposed by: Alexander Duncan

Solution. $\boxed{64}$

Note that $\sqrt{(c-a)^2 + (d-b)^2}$ is the distance between a point on the circle with radius 2 and center at $(0,0)$ and a point on the circle with radius 3 and center $(5,12)$. The minimum distance occurs when these points lie on the line $y = \frac{12}{5}x$ and on the region of the circle that is closest to the other point, and this distance is $13-2-3=8$, so the maximum value of $(c-a)^2 + (d-b)^2$ is $8^2 = \boxed{64}$. □

______________ 17. **[24]** Ethan and 2 girls are placed independently and uniformly at random on the circumference of a circle of radius 1. Ethan can attract any girl within a distance of 1, but the girls are jealous of each other if the distance between them is at most 1. Find the probability that Ethan can attract both girls without making them jealous.

Proposed by: Jerry Xu

Solution. $\boxed{\dfrac{1}{36}}$

Fix Ethan. The first girl has a $\frac{1}{3}$ probability of being within attracting distance to Ethan. The second girl's probability of being attracted to Ethan without being jealous of the first girl is equally distributed from 0 to $\frac{1}{6}$. Our answer is $\frac{1}{3} \cdot \frac{1}{12} = \boxed{\dfrac{1}{36}}$. □

______________ 18. **[24]** Let $a > b > c > 0$ be real numbers satisfying

$$(a-b)(a-c) + 7 = b^2 + c^2$$
$$(b-a)(b-c) + 31 = a^2 + c^2$$
$$(c-a)(c-b) + 31 = a^2 + b^2$$

Find (a, b, c).

Proposed by: Samuel Tsui

Solution. $\boxed{(5,3,2)}$

Subtracting pairwise equations gives $(a-b)(a+b-c) = 12$, $(a-c)(a+c-b) = 12$, $(b-c)(b+c-a) = 0$. Thus we have $a = b+c$ and plugging this into the first equation gives $bc = 6$. Now solving we get $a = \pm 5$, $b = \pm 3, \pm 2$, $c = \pm 2, \pm 3$ so the answer is $\boxed{(5,3,2)}$. □

LMT Fall 2025 Guts Round Solutions- Part 7

Team Name:

______________ 19. **[26]** Awashonks is graphing the function

$$f(x) = \left\lfloor \frac{5}{3}x \right\rfloor + \left\lceil \frac{5}{9}x^2 \right\rceil + \left\lfloor \frac{5}{27}x^3 \right\rfloor + \left\lceil \frac{5}{81}x^4 \right\rceil$$

from $x = 0$ to $x = 6$. Find the minimum distance her pen must travel, considering any jumps it must take.

Proposed by: Edwin Zhao

Solution. $\boxed{156}$

Notice since everything is floored, her pen can only move horizontally or vertically. Additionally, her pen can only move to the right or upwards. The horizontal movement her pen takes is equal to 6 and the vertical movement her pen takes is equal to plugging in 6 to the expression. Thus, our answer is $6+10+20+40+80=\boxed{156}$. □

_______________ 20. **[26]** Let ABC be a triangle. From each midpoint of an edge of ABC draw an altitude to each of the two sides that it does not lie on. This forms a hexagon in the interior of ABC. Given that $AB=5$, $BC=6$, and $AC=7$, find the area of this hexagon.

Proposed by: Henry Eriksson

Solution. $\boxed{3\sqrt{6}}$

We will find the area outside of the hexagon instead.

Let the midpoints be D, E, and F, labelled in the standard order. Let X be the intersection of the altitude from F to AC and the altitude from E to AB. Define Y and Z similarly to be the intersection of the altitudes near vertices B and C, respectively.

Note that AEF, FDB, and ECD are all ABC scaled down by a factor of 2, and therefore are congruent and have area $\frac{3}{2}\sqrt{6}$ (because $A_{ABC}=\sqrt{9\cdot4\cdot3\cdot2}=6\sqrt{6}$ by Heron's Formula).

Then $AXE\cong FYD$ because they are the same construction in congruent triangles. Therefore we can delete AXE and change the quadrilateral using the vertex B into the triangle FDB, which we know has area $\frac{3}{2}\sqrt{6}$. Similarly, move AXF to complete triangle ECD, also with area $\frac{3}{2}\sqrt{6}$. Now we have calculated all of the area outside of the hexagon, and found it to be $\frac{3}{2}\sqrt{6}+\frac{3}{2}\sqrt{6}=3\sqrt{6}$.

This means that the area of the hexagon is $6\sqrt{6}-3\sqrt{6}=\boxed{3\sqrt{6}}$. □

_______________ 21. **[26]** Let (a,b,c) be a ordered triple of positive reals that satisfy

$$\lfloor ab\rfloor c=3$$
$$\lfloor bc\rfloor a=4$$
$$\lfloor ca\rfloor b=5.$$

Find the sum of all possible values of a.

Proposed by: Samuel Tsui

Solution. $\boxed{\frac{22}{3}}$

Let $m=\lfloor ab\rfloor$, $n=\lfloor bc\rfloor$, $p=\lfloor ca\rfloor$ where we know m,n,p are positive integers. Rewriting our equations we have $m=\left\lfloor\frac{20}{np}\right\rfloor$, $n=\left\lfloor\frac{15}{pm}\right\rfloor$, $p=\left\lfloor\frac{12}{mn}\right\rfloor$ meaning $np\le20$, $pm\le15$, $mn\le12$. Quickly, we find $m=1$ and testing a few cases the only possible values are $(n,p)=(1,12),(2,6),(3,4)$ which yields $(a,b,c)=\left(4,\frac{5}{12},3\right)\left(2,\frac{5}{6},3\right),\left(\frac{4}{3},\frac{5}{4},3\right)$, giving an answer of $4+2+\frac{4}{3}=\boxed{\frac{22}{3}}$. □

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____________ 22. **[30]** Let $ABCD$ be a trapezoid such that $AB \parallel CD$ and $DA = AB = BC = 5$. Let $EFGH$ be a square of side length 1 inside $ABCD$ such that GH lies on CD with G in between C and H. If $BF = 5$ and $CG = 7$, find the area of ADE.

Proposed by: Samuel Tsui

Solution. $\boxed{\frac{9}{2}}$

Observe that $BC^2 + BF^2 = GC^2 + GF^2$ so $\angle CBF = 90^\circ$. Thus we have $\sin(\angle BCG) = \sin(\angle BCF + \angle FCG) = \frac{\sqrt{2}}{2} \cdot \frac{7}{5\sqrt{2}} + \frac{\sqrt{2}}{2} \cdot \frac{1}{5\sqrt{2}} = \frac{4}{5}$. This implies $AH \perp CD$, $AH = 4$ so $AE = 3$, and $DH = 3$ yielding the answer $\frac{1}{2} \cdot 3 \cdot 3 = \boxed{\frac{9}{2}}$. □

____________ 23. **[30]** On Alfredo's strange clock, the minute and hour hand are identical. During a 12 hour period, find the number of times for which it is impossible for Alfredo to tell the real time. (For example, if the two hands are pointed at 12 and 3, Alfredo can tell that the real time is 3:00.)

Proposed by: Ben

Solution. $\boxed{132}$

Let α, β be the two angles of the hands from the 12 on the clock (For example, a hand pointing at 1 would have angle $\theta = 30^\circ$ from the 12). Now, take α to be the angle of the hour hand. Then, if the hour hand is in between the numbers N and $N+1$ on the clock, then we can express α as

$$\alpha = 30N + c$$

where $0 \le c < 30$. Then, in order for the two hands to form a valid time, we must have $\beta = 12c$. In other words, we have

$$12\alpha \equiv \beta \pmod{360}$$

So, $12\alpha \equiv \beta \pmod{360}$ is the condition needed for α as the hour hand and β as the minute hand to be a valid time.

So, Alfredo can't tell the time when both the conditions $12\alpha \equiv \beta \pmod{360}$ and $12\beta \equiv \alpha \pmod{360}$ are met. We have

$$12(12\alpha) \equiv 12(\beta) \equiv \alpha \pmod{360}$$

$$143\alpha \equiv 0 \pmod{360}$$

So, we have 143 solutions for α, however, whenever $\alpha = \beta$ Alfredo can tell the time. For these cases, we must have $\alpha = \beta = 12\alpha \pmod{360} \implies 11\alpha \equiv 0 \pmod{360}$. There are 11 such α. Therefore, our final answer is $143 - 11 = \boxed{132}$. □

____________ 24. **[30]** For an integer with at least three digits define its *smae* to be the integer formed by swapping the second and third digit. For example, the smae of 12345 is 13245. Let n be a 5-digit positive integer, and let m be its smae. Suppose n is the unique such 5-digit integer whose prime factorization is the first three digits of m times the last three digits of m. Find n.

Proposed by: Ryan Tang

Solution. $\boxed{11009}$

Define $\overline{a_n a_{n-1} \dots a_2 a_1 a_0} = 10^n a_n + \cdots + 10^1 a_1 + 10^0 a_0$.

Let $n = \overline{abcde}$. Thus, our integer satisfies the condition $n = \overline{acb} \cdot \overline{bde}$. In the multiplication, we have that the unit digit is $be \equiv e \pmod{10}$. However, since $\overline{bde}$ must be prime, we have that $e \equiv 1 \pmod{2}$, and so $b = 1$.

We now have $n = \overline{ac1} \cdot \overline{1de}$. Taking modulo 100, we see that $n \equiv (10c+1)(10d+e) \equiv 10d+e+10ce \pmod{100}$. However, we also know that $n \equiv 10d+e \pmod{100}$. Hence, $10ce \equiv 0 \pmod{100}$. Clearly, this implies that $c = 0$ by same reasoning. Hence, $n = \overline{a01} \cdot \overline{1de} = \overline{a10de}$. Expanding n, we get $n = \overline{1de} + a \cdot \overline{1de00}$. We have that $ae+1 \equiv 0 \pmod{10}$. We also have that $ad < 10$ since ad can't have any carry. Thus, either $d = 0$ OR one of a, d is 1. If $d = 0$, $e = 1, 3, 7, 9$. Using $ae+1 \equiv 0 \pmod{10}$, it can be checked that $a = 1, e = 9$ is the only valid answer. Since we are given the answer is unique, $\boxed{11009}$, is the only valid answer. □

LMT Fall 2025 Guts Round Solutions- Part 9

Team Name: ______________________________

__________ 25. **[30]** Find the number of primes p which can be written in the form $2^n + 2^m + 1$ for positive integers $m \le n < 100$. If your answer is A and the correct answer is E, then you will earn $\max(0, \lceil 30 - 0.45|E - A| \rceil)$ points.

Proposed by: Jerry Xu

Solution. $\boxed{282}$

Fix n. For any given m, the probability that $2^n + 2^m + 1$ is prime can be approximated as $\frac{2}{\ln 2^n}$ by the prime number theorem (where we note $2^n + 2^m + 1$ is always odd, so the probability is doubled). Therefore, a rough estimate for the expected number of valid m for any given n is $\frac{2n}{\ln 2^n} = \frac{2n}{n \ln 2} = \frac{2}{\ln 2}$ (since there are n possible values of m). It is known that $\ln 2 \approx 0.7$, so

$$\sum_{n=1}^{99} \frac{2n}{\log 2^n} \approx \frac{2}{0.7} \cdot 99 \approx 283.$$

This gives a full 30 points. □

__________ 26. **[30]** The numbers $1, 2, \ldots, 10$ are placed in a line in that order. Then, for each of the nine pairs of consecutive integers, an operation is chosen independently at uniformly at random from $+, -, \cdot, \div$ and placed between those integers. For example, one possible outcome for the resulting expression is

$$1 + 2 \cdot 3 + 4 \div 5 \cdot 6 \cdot 7 + 8 - 9 \div 10.$$

Find the expected value of the resulting expression. If the true answer is A and your answer is E, you will earn $\max(0, \lceil 30 - 0.45|E - A|^2 \rceil)$ points.

Proposed by: Jerry Xu

Solution. $\boxed{32.9842}$

Group terms by consecutive × and ÷ symbols under standard PEMDAS. For example, the terms of

$$1 + 2 \cdot 3 + 4 \div 5 \cdot 6 \cdot 7 + 8 - 9 \div 10$$

would be $1, 2 \cdot 3, 4 \div 5 \cdot 6 \cdot 7, 8, 9 \div 10$. Note that for each term after the first (the one containing 1), it is equally likely to be added or subtracted in the final expression, so it's expected value is 0. Therefore, it suffices to compute the expected value of the first term.

The first term has n symbols if and only if the first $n-1$ symbols are all × or ÷, which occurs with probability $\frac{2}{4}$, and then next symbol is + or −, which also occurs with probability $\frac{2}{4}$. (If $n = 10$, then there is no next symbol.) Each symbol k can either multiply or divide, so they contribute a factor of $\frac{k+\frac{1}{k}}{2}$. Since we know the symbols appear as $1, 2, \ldots, 10$ in that order, our answer is

$$\sum_{n=1}^{9} \left(\frac{2}{4}\right)^n \prod_{k=1}^{n} \left(\frac{k+\frac{1}{k}}{2}\right) + \left(\frac{2}{4}\right)^9 \prod_{k=1}^{10} \left(\frac{k+\frac{1}{k}}{2}\right).$$

Define $a_n = \left(\frac{2}{4}\right)^n \prod_{k=1}^{n}\left(\frac{k+\frac{1}{k}}{2}\right)$. First, we have $a_1 = \frac{2}{4}\cdot\frac{1+\frac{1}{1}}{2} = \frac{1}{2}$. Then, $a_n = a_{n-1}\cdot\frac{k+\frac{1}{k}}{4} \approx a_{n-1}\cdot\frac{k}{4}$, so $a_n \approx a_1 \cdot \frac{n!}{4^{n-1}}$. Therefore, we can estimate

$$\sum_{n=1}^{9} a_n \approx \frac{1}{2}\left(1+\frac{2!}{4}+\frac{3!}{16}+\cdots+\frac{9!}{4^8}\right) \approx 6.3.$$

To account for our underestimation, we can scale back by $\frac{2+\frac{1}{2}}{2}\cdot\frac{3+\frac{1}{3}}{3}\cdot\frac{4+\frac{1}{4}}{4} \approx 1.5$, giving $6.3\cdot 1.5 \approx 9.5$. Furthermore,

$$\left(\frac{2}{4}\right)^9 \prod_{k=1}^{10}\left(\frac{k+\frac{1}{k}}{2}\right) \approx 2a_{10} \approx 2\cdot\frac{1}{2}\cdot\frac{10!}{4^9} \approx 13.8.$$

Again, scaling back gives $13.8\cdot 1.5 \approx 20.7$. Therefore, our answer is estimated to be around 30.2. This gives 27 points. □

__________ 27. **[30]** N points are chosen independently and uniformly at random on a circle of radius 1. Find the smallest positive integer N such that the expected value of the area of the N-gon formed by those N points is at least 3.14. If the correct answer is E and your answer is A, you will earn $\max(0, \lceil 30 - 0.45|E-A|\rceil)$ points.

Proposed by: Jerry Xu

Solution. $\boxed{278}$

Let the polygon be $P = P_1P_2\dots P_N$, and let the center of the unit circle be O. Let $\angle P_iOP_{i+1} = \theta_i$, so that $\theta_1,\dots,\theta_N$ are chosen independently uniformly at random from $(0,2\pi)$ such that $\theta_1+\cdots+\theta_N = 2\pi$. By law of sines, we have

$$\mathbb{E}[P] = \mathbb{E}\left(\sum_{i=1}^{N}[P_iOP_{i+1}]\right) = \mathbb{E}\left(\sum_{i=1}^{N}\frac{1}{2}1^2\sin(\theta_i)\right) = \frac{N}{2}\mathbb{E}[\sin(\theta_1)].$$

We utilize the Taylor series for sin. In particular, we know that for small x, $\sin(x) \approx x - \frac{x^3}{6}$ (where the $\frac{x^5}{120}$ term is too small to matter). Then,

$$\mathbb{E}[\sin(\theta_1)] \approx \mathbb{E}[\theta_1] - \frac{1}{6}\mathbb{E}[\theta_1^3].$$

Since $\sum_{i=1}^{N}\theta_i = 2\pi$, we know $\mathbb{E}[\theta_1] = \frac{2\pi}{N}$. Then,

$$\frac{N}{2}\mathbb{E}[\sin(\theta_1)] = \frac{N}{2}\left(\frac{2\pi}{N} - \mathbb{E}[\theta_1^3]\right) = \pi - \frac{N}{12}\mathbb{E}[\theta_1^3].$$

Let $X = \frac{\theta_1}{2\pi}$. We compute

$$\mathbb{E}[X^3] = 3\int_0^1 x^{3-1}\Pr(X>x)\,dx = 3\int_0^1 x^{3-1}(1-x)^{N-1}\,dx = \frac{6}{N(N+1)(N+2)}.$$

This is because $\Pr(X>x)$ is the probability that the angle between P_1 and P_2 is at least $2\pi x$, which occurs if and only if all $P_2, P_3, \dots, P_N$ lie after the arc of length $2\pi x$. Then,

$$\mathbb{E}[\theta_1^3] = \frac{6(2\pi)^3}{N(N+1)(N+2)} = \frac{48\pi^3}{N(N+1)(N+2)}.$$

We want $\frac{N}{12}\mathbb{E}[\theta_1^3] = \frac{4\pi^3}{(N+1)(N+2)} < \pi - 3.14 \approx 0.0016$. $24\pi^3 \approx 4\cdot 10\cdot\pi \approx 124$. Then, we want

$$(N+1)(N+2) \geq \frac{124}{0.0016} = 77500.$$

This gives $N \approx 277$, giving a full 30 points.

□