LMT Fall 2023 Guts Round - Part 1

Team Name: ____________________

____________ 1. **[12]** Calculate

$$(4!-5!+2^5+2^6)\cdot\frac{12!}{7!}+(1-3)(4!-2^4).$$

____________ 2. **[12]** The expression $\sqrt{9!+10!+11!}$ can be expressed as $a\sqrt{b}$ for positive integers a and b, where b is squarefree. Find a.

____________ 3. **[12]** For real numbers a and b, $f(x)=ax^{10}-bx^4+6x+10$ for all real x. Given that $f(42)=11$, find $f(-42)$.

LMT Fall 2023 Guts Round - Part 2

Team Name: ____________________

____________ 4. **[15]** How many positive integers less than or equal to 2023 are divisible by 20, 23, or both?

____________ 5. **[15]** Larry the ant crawls along the surface of a cylinder with height 48 and base radius $\frac{14}{\pi}$. He starts at point A and crawls to point B, traveling the shortest distance possible. What is the maximum this distance could be?

____________ 6. **[15]** For a given positive integer n, Ben knows that $\lfloor 20x\rfloor=n$, where x is real. With that information, Ben determines that there are 3 distinct possible values for $\lfloor 23x\rfloor$. Find the least possible value of n.

LMT Fall 2023 Guts Round - Part 3

Team Name: ____________________

____________ 7. **[18]** Let ABC be a triangle with area 1. Points D, E, and F lie in the interior of $\triangle ABC$ in such a way that D is the midpoint of AE, E is the midpoint of BF, and F is the midpoint of CD. Compute the area of DEF.

____________ 8. **[18]** Edwin and Amelia decide to settle an argument by running a race against each other. The starting line is at a given vertex of a regular octahedron and the finish line is at the opposite vertex. Edwin has the ability to run straight through the octahedron, while Amelia must stay on the surface of the octahedron. Given that they tie, what is the ratio of Edwin's speed to Amelia's speed?

____________ 9. **[18]** Jxu is rolling a fair three-sided die with faces labeled 0, 1, and 2. He keeps going until he rolls a 1, immediately followed by a 2. What is the expected number of rolls Jxu makes?

LMT Fall 2023 Guts Round - Part 4

Team Name: ____________________

____________ 10. **[21]** For real numbers x and y, $x+xy=10$ and $y+xy=6$. Find the sum of all possible values of $\frac{x}{y}$.

____________ 11. **[21]** Derek is thinking of an odd two-digit integer n. He tells Aidan that n is a perfect power and the product of the digits of n is also a perfect power. Find the sum of all possible values of n.

____________ 12. **[21]** Let a three-digit positive integer $N=\overline{abc}$ (in base 10) be *stretchable* with respect to m if N is divisible by m, and when N's middle digit is duplicated an arbitrary number of times, it's still divisible by m. How many three-digit positive integers are *stretchable* with respect to 11? (For example, 432 is *stretchable* with respect to 6 because $433\ldots32$ is divisible by 6 for any positive integer number of 3s.)

LMT Fall 2023 Guts Round - Part 5

Team Name: ______________________________

______________ 13. **[27]** How many trailing zeroes are in the base-2023 expansion of 2023!?

______________ 14. **[27]** The three-digit positive integer $k = \overline{abc}$ (in base 10, with a nonzero) satisfies

$$\overline{abc} = c^{2ab-1}.$$

Find the sum of all possible k.

______________ 15. **[27]** For any positive integer k, let a_k be defined as the greatest nonnegative real number such that in an infinite grid of unit squares, no circle with radius less than or equal to a_k can partially cover at least k distinct unit squares. (A circle partially covers a unit square only if their intersection has positive area.)

Find the sum of all positive integers $n \le 12$ such that $a_n \ne a_{n+1}$.

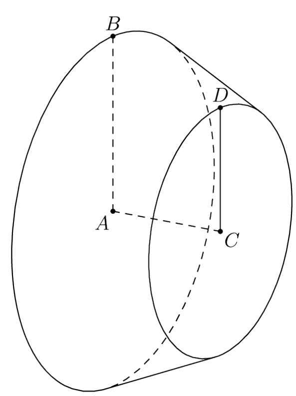
LMT Fall 2023 Guts Round - Part 6

Team Name: ______________________________

______________ 16. **[33]** Let $p(x)$ and $q(x)$ be polynomials with integer coefficients satisfying $p(1) = q(1)$. Find the greatest integer n such that $\frac{p(2023)-q(2023)}{n}$ is an integer no matter what $p(x)$ and $q(x)$ are.

______________ 17. **[33]** Find all ordered pairs of integers (m,n) that satisfy $n^3 + m^3 + 231 = n^2m^2 + nm$.

______________ 18. **[33]** Ben rolls the frustum-shaped piece of candy (shown below) in such a way that the lateral area is always in contact with the table. He rolls the candy until it returns to its original position and orientation.



Given that $AB = 4$ and $BD = CD = 3$, find the length of the path traced by A.

LMT Fall 2023 Guts Round - Part 7

Team Name: ______________________________

______________ 19. **[39]** In their science class, Adam, Chris, Eddie and Sam are independently and randomly assigned an integer grade between 70 and 79 inclusive. Given that they each have a distinct grade, what is the expected value of the maximum grade among their four grades?

______________ 20. **[39]** Let $ABCD$ be a regular tetrahedron with side length 2. Let point E be the foot of the perpendicular from D to the plane containing $\triangle ABC$. There exist two distinct spheres ω_1 and ω_2, centered at points O_1 and O_2 respectively, such that both O_1 and O_2 lie on $\overrightarrow{DE}$ and both spheres are tangent to all four of the planes ABC, BCD, CDA, and DAB. Find the sum of the volumes of ω_1 and ω_2.

______________ 21. **[39]** Evaluate

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(i+j+k+1)2^{i+j+k+1}}.$$

LMT Fall 2023 Guts Round - Part 8

Team Name: ______________________________

______ 22. **[45]** In $\triangle ABC$, let I_A, I_B, and I_C denote the A, B, and C-excenters, respectively. Given that $AB = 15$, $BC = 14$ and $CA = 13$, find $\frac{[I_A I_B I_C]}{[ABC]}$.

______ 23. **[45]** The polynomial

$$x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 6x^6 + 5x^7 + 4x^8 + 3x^9 + 2x^{10} + x^{11}$$

has distinct complex roots $z_1, z_2, \ldots, z_n$. Find

$$\sum_{k=1}^{n} \left|\Re(z_n^2)\right| + \left|\Im(z_n^2)\right|,$$

where $\Re z$ and $\Im z$ indicate the real and imaginary parts of z, respectively. Express your answer in simplest radical form.

______ 24. **[45]** Given that $\sin 33^\circ + 2\sin 161^\circ \cdot \sin 38^\circ = \sin n^\circ$, compute the least positive integer value of n.

LMT Fall 2023 Guts Round - Part 9

Team Name: ______________________________

______ 25. **[30]** Submit a prime between 2 and 2023, inclusive. If you don't, or if you submit the same number as another team's submission, you will receive 0 points. Otherwise, your score will be

$$\min(30, \lfloor 4 \cdot \ln(x) \rfloor),$$

where x is the positive difference between your submission and the closest valid submission made by another team.

______ 26. **[30]** Sam, Derek, Jacob, and Muztaba are eating a very large pizza with 2023 slices. Due to dietary preferences, Sam will only eat an even number of slices, Derek will only eat a multiple of 3 slices, Jacob will only eat a multiple of 5 slices, and Muztaba will only eat a multiple of 7 slices. How many ways are there for Sam, Derek, Jacob, and Muztaba to eat the pizza, given that all slices are identical and order of slices eaten is irrelevant? If your answer is A and the correct answer is C, the number of points you receive will be:

$$\max\left(0, \left\lfloor 30\left(1 - 2\sqrt{\frac{|A-C|}{C}}\right)\right\rfloor\right)$$

______ 27. **[30]** Let $\Omega(k)$ denote the number of perfect square divisors of k. Compute

$$\sum_{k=1}^{10000} \Omega(k).$$

If your answer is A and the correct answer is C, the number of points you recieve will be

$$\max\left(0, \left\lfloor 30\left(1 - 4\sqrt{\frac{|A-C|}{C}}\right)\right\rfloor\right)$$