Team Round

Lexington High School

May 14th, 2022

1. **[7]** Derek and Jacob have a cake in the shape a rectangle with dimensions 14 inches by 9 inches. They make a deal to split it: Derek takes home the portion of the cake that is less than one inch from the border, while Jacob takes home the remainder of the cake. Let $D : J$ be the ratio of the amount of cake Derek took to the amount of cake Jacob took, where D and J are relatively prime positive integers. Find $D + J$.

   Proposed by Kevin Zhao

   Solution. $\boxed{5}$

   Note that Jacob takes home the inner $12 \times 7 = 84$ square inches, while Derek takes home the rest. The total area is $14 \cdot 9 = 126$, so Derek also takes home 42 square units. The ratio is $1 : 2$, and $1 + 2 = \boxed{3}$. □

2. **[7]** Five people are standing in a straight line, and the distance between any two people is a unique positive integer number of units. Find the least possible distance between the leftmost and rightmost people in the line in units.

   Proposed by Ephram Chun

   Solution. $\boxed{11}$

   By Inspection $3, 1, 5, 2$ works giving the lengths $1, 2, 3, 4, 5, 6, 7, 8, 9, 11$. Thus, our answer is $3 + 1 + 5 + 2 = \boxed{11}$ □

3. **[7]** Let the four real solutions to the equation $x^2 + \frac{144}{x^2} = 25$ be r_1, r_2, r_3, and r_4. Find $|r_1| + |r_2| + |r_3| + |r_4|$.

   Proposed by Kevin Zhao

   Solution. $\boxed{14}$

   The solutions are $\pm 3, \pm 4$. □

4. **[7]** Jeff has a deck of 12 cards: 4 Ls, 4 Ms, and 4 Ts. Armaan randomly draws three cards without replacement. The probability that he takes 3 Ls can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

   Proposed by Kevin Zhao

   Solution. $\boxed{56}$

   The probability is $\frac{4}{12} \cdot \frac{3}{11} \cdot \frac{2}{10} = \frac{1}{55}$ so the answer is $1 + 55 = \boxed{56}$. □

5. **[7]** Find the sum

$$\sum_{n=1}^{2020} \gcd(n^3 - 2n^2 + 2021, n^2 - 3n + 3).$$

   Proposed by Ephram Chun

   Solution. $\boxed{2020}$

   Claim: I claim that the answer is $\boxed{2020}$

   Proof: We proceed with the Euclidean Algorithm. $\gcd[a^3 - 2a^2 + 2021, a^2 - 3a + 3]$ $\gcd[2018, a^2 - 3a + 3]$ Thus we see that no matter what value a is the $a^2 - 3a + 3$ is always going to be odd through a parity argument. So, the $\gcd[a^3 - 2a^2 + 4, a^2 - 3a + 3]$ will always be 1. Thus, our answer is simply $2020 * 1 = \boxed{2020}$ □

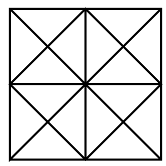
6. **[7]** For all y, define cubic $f_y(x)$ such that $f_y(0) = y$, $f_y(1) = y+12$, $f_y(2) = 3y^2$, $f_y(3) = 2y+4$. For all y, $f_y(4)$ can be expressed in the form $ay^2 + by + c$ where a, b, c are integers. Find $a+b+c$.

   Proposed by Ella Kim

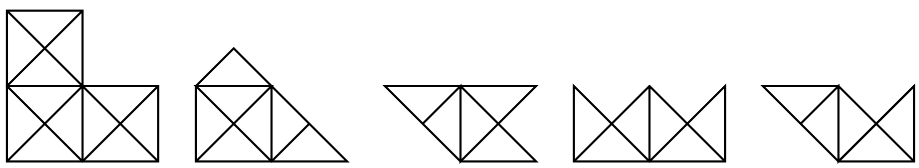
   Solution. $\boxed{57}$

   Note that here, we can let $y = 1$. Thus, the values of f are 1, 13, 3, 6; differences are 12, -10, 3; Second differences are -22, 13, so since second differences here are arithmetically increasing, the next term of second difference is 48; difference $48+6 = 54$ and constant $54+3 = \boxed{57}$. □

7. **[7]** Kevin has a square piece of paper with creases drawn to split the paper in half in both directions, and then each of the four small formed squares diagonal creases drawn, as shown below.



   Find the sum of the corresponding numerical values of figures below that Kevin can create by folding the above piece of paper along the creases. (The figures are to scale.) Kevin cannot cut the paper or rip it in any way.



1 2 4 8 16

   Proposed by Kevin Zhao

   Solution. $\boxed{22}$

   Kevin can create the second, third, and fifth configurations. The sum is $2+4+16 = \boxed{22}$. □

8. **[7]** The 53-digit number

$$37{,}984{,}318{,}966{,}591{,}152{,}105{,}649{,}545{,}470{,}741{,}788{,}308{,}402{,}068{,}827{,}142{,}719$$

   can be expressed as n^{21} where n is a positive integer. Find n.

   Proposed by Kevin Zhao

   Solution. $\boxed{319}$

   Note that by Fermat's Little Theorem, $n^2 1 \equiv \pm n \pmod{100}$ for odd numbers. Thus, we see that this number has 53 digits and is around $3.79 \cdot 10^{52}$. Noting that

$$300^{21} = 10^{42} \cdot 3^{21} \leq 3 \cdot 10^{52} \leq 3.79 \cdot 10^{52}$$

   and $350^{21} = 10^{42} \cdot 3.5^{21}$ shows that since $3.5^2 = 12.25$, then $3.5^6 > 1728$ and $3.5^7 \geq 5000$ so $3.5^{21} \geq 1.25 \cdot 10^{11}$, and so $350^{21} \geq 1.25 \cdot 10^{53} > 3.79 \cdot 10^{52}$. Thus, our number is between 300 and 350, meaning it can only possibly be $\boxed{319}$. □

9. **[7]** Let $r_1, r_2, \ldots, r_{2021}$ be the not necessarily real and not necessarily distinct roots of $x^{2022} + 2021x = 2022$. Let $S_i = r_i^{2021} + 2022r_i$ for all $1 \leq i \leq 2021$. Find $\left|\sum_{i=1}^{2021} S_i\right| = |S_1 + S_2 + \ldots + S_{2021}|$.

   Proposed by Kevin Zhao

Solution. $\boxed{4082420}$

Note that

$$S_i = r_i^{2021} + 2022r_i = \frac{r_i^{2022} + 2021r_i}{r_i} - 2021 + 2022r_i = \frac{2022}{r_i} - 2021 + 2022r_i.$$

Since from Vieta's, the sum of roots is zero and the sum of all values of r_i is $\frac{2021}{2022}$, then our sum is

$$-2021 \cdot 2021 + \frac{2021}{2022} \cdot 2022 = -2021 \cdot 2020 = -\boxed{4082420}.$$

□

10. **[7]** In a country with 5 distinct cities, there may or may not be a road between each pair of cities. It's possible to get from any city to any other city through a series of roads, but there is no set of three cities $\{A, B, C\}$ such that there are roads between A and B, B and C, and C and A. How many road systems between the five cities are possible?

Proposed by Kevin Zhao

Solution. $\boxed{167}$

Note that the condition means that either a pentagon of connected vertices exists, or the graph is bipartite. If a pentagon of connected vertices exists, we note that WLOG letting $A - B - C - D - E - A$ be the cycle, we cannot add any more edges to the graph. Thus, this scenario has, fixing A, $\binom{4}{2} \cdot 2 = 12$ ways. In the bipartite case, we have a tree or a 4-cycle. A 4-cycle means that no other pair of vertices in the 4-cycle are connected, and only opposites can. If we have multiple 4-cycles, then we have $\binom{5}{2} = 10$ ways to choose the two such edges. If we have one 4-cycle, we have $5 \cdot 4 = 20$ ways. If we have no 4-cycles, then we have a tree which is just partitioning. Vertex orders can be $4-1-1-1-1$, $3-2-1-1-1$, $2-2-2-1-1$. The first case gets 5 ways, the second gets $5 \cdot 4 \cdot 3 = 60$ ways, the third gets $\frac{5!}{2} = 60$. The sum is $12 + 10 + 20 + 5 + 60 + 60 = \boxed{167}$. □