Speed Round

Lexington High School

May 14th, 2022

1. **[6]** Aidan walks into a skyscraper's first floor lobby and takes the elevator up 50 floors. After exiting the elevator, he takes the stairs up another 10 floors, then takes the elevator down 30 floors. Find the floor number Aidan is currently on.

   Proposed by Kevin Zhao

   Solution. $\boxed{31}$

   $1+50+10-30=\boxed{31}$. □

2. **[6]** Jeff flips a fair coin twice and Kaylee rolls a standard 6-sided die. The probability that Jeff flips 2 heads and Kaylee rolls a 4 is P. Find $\frac{1}{P}$.

   Proposed by Kevin Zhao

   Solution. $\boxed{24}$

   The number of possibilities is $2\cdot 2\cdot 6=\boxed{24}$, with one working. □

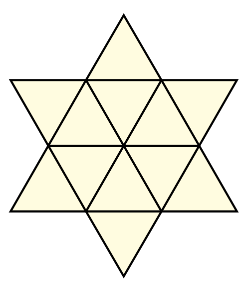
3. **[6]** Given that $a\odot b=a+\frac{a}{b}$, find $(4\odot 2)\odot 3$.

   Proposed by Kevin Zhao

   Solution. $\boxed{8}$

   $(4\odot 2)\odot 3=6\odot 3=\boxed{8}$. □

4. **[6]** The following star is created by gluing together twelve equilateral triangles each of side length 3. Find the outer perimeter of the star.

   Proposed by Kevin Zhao

   Solution. $\boxed{36}$

   The star has 12 sides each of length 3. $12\cdot 3=\boxed{36}$. □

5. **[6]** In Lexington High School's Math Team, there are 40 students: 20 of whom do science bowl and 22 of whom who do LexMACS. What is the least possible number of students who do both science bowl and LexMACS?

   Proposed by Kevin Zhao

Solution. $\boxed{2}$

The total number of students is equal to the sum of those who do none, one, and two clubs. Hence, we maximize when nobody does no clubs and $\boxed{2}$ people do both clubs. □

6. **[6]** What is the least positive integer multiple of 3 whose digits consist of only 0s and 1s? The number does not need to have both digits.

Proposed by Kevin Zhao

Solution. $\boxed{111}$

In order for divisibility by 3, the sum of digits must be divisible by 3. $\boxed{111}$ is the smallest value that works. □

7. **[6]** Two fair 6-sided dice are rolled. The probability that the product of the numbers rolled is at least 30 can be written as $\frac{a}{b}$ where a and b are relatively prime positive integers. Find $a+b$.

Proposed by Kevin Zhao

Solution. $\boxed{13}$

There are 3 such ways: $(5,6)$, $(6,5)$, and $(6,6)$. $\frac{3}{6^2} = \frac{1}{12}$ and $12+1 = \boxed{13}$. □

8. **[6]** At the LHS Math Team Store, 5 hoodies and 1 jacket cost \$13, and 5 jackets and 1 hoodie cost \$17. Find how much 15 jackets and 15 hoodies cost, in dollars.

Proposed by Boyan Litchev

Solution. $\boxed{75}$

6 jackets and 6 hoodies cost \$30, so 15 jackets and 15 hoodies cost $\frac{15}{6} \cdot 30 = \boxed{75}$. □

9. **[6]** Eric wants to eat ice cream. He can choose between 3 options of spherical ice cream scoops. The first option consists of 4 scoops each with a radius of 3 inches, which costs a total of \$3. The second option consists of a scoop with radius 4 inches, which costs a total of \$2. The third option consists of 5 scoops each with diameter 2 inches, which costs a total of \$1. The greatest possible ratio of volume to cost of ice cream Eric can buy is $n\pi$ cubic inches per dollar. Find $3n$.

Proposed by Kevin Zhao

Solution. $\boxed{144}$

The first option gives a unit price of $4 \cdot \frac{4}{3} \cdot \pi \cdot 3^3 \div 3 = 48\pi$. The second option gives a unit price of $1 \cdot \frac{4}{3} \cdot \pi \cdot 4^3 \div 2 = \frac{128}{3}\pi < 48\pi$. The third option has a radius of $\frac{2}{2} = 1$, so this gives a unit price of $5 \cdot \frac{4}{3} \cdot \pi \cdot 1^3 \div 1 = \frac{20}{3}\pi < 48\pi$, so our answer is $48 \cdot 3 = \boxed{144}$. □

10. **[6]** Jen claims that she has lived during at least part of each of five decades. What is the least possible age that Jen could be? (Assume that age is always rounded down to the nearest integer.)

Proposed by Kira Tang

Solution. $\boxed{30}$

Suppose that Jen has lived in the 1970s, 1980s, 1990s, 2000s, and 2010s. To minimize her age, assume that Jen was born on December 31, 1979 and that today's date is January 1, 2010. In this case, Jen's last birthday was December 31, 2009. Thus, Jen's least possible age is $2009 - 1979 = \boxed{30}$. □

11. **[6]** A positive integer n is called *Maisylike* if and only if n has fewer factors than $n-1$. Find the sum of the values of n that are *Maisylike*, between 2 and 10, inclusive.

Proposed by Ephram Chun

Solution. $\boxed{21}$

Listing all of the values, we see that 2 has two factors, 3 has two, 4 has three, 5 has two, 6 has four, 7 has two, 8 has four, 9 has three, and 10 has four. By inspection, we see that $n = 5$, $n = 7$, and $n = 9$ are *Maisylike*. Then, the desired sum is $5+7+9 = \boxed{21}$ □

12. **[6]** When Ginny goes to the nearby boba shop, there is a 30% chance that the employee gets her drink order wrong. If the drink she receives is not the one she ordered, there is a 60% chance that she will drink it anyways. Given that Ginny drank a milk tea today, the probability she ordered it can be written as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find the value of $a+b$.

    Proposed by Kira Tang

    Solution. $\boxed{79}$

    There is a 70% chance that the guy at the counter gets Ginny's order right (we assume she will drink a correct order 100% of the time). Meanwhile, the probability that she gets an incorrect drink and still drinks it is $\frac{3}{10}\cdot\frac{3}{5} = \frac{9}{50}$.

    Thus, the probability that she got a correct order given that she drank the beverage is $\frac{\frac{7}{10}}{\frac{9}{50}+\frac{7}{10}} = \frac{35}{44}$. Thus, $a+b = \boxed{79}$. □

13. **[6]** Alex selects an integer m between 1 and 100, inclusive. He notices there are the same number of multiples of 5 as multiples of 7 between m and $m+9$, inclusive. Find how many numbers Alex could have picked.

    Proposed by Muztaba Syed

    Solution. $\boxed{42}$

    Notice that there are 10 values from m to $m+9$ inclusive, meaning there must be 2 multiples of 5. This means that there are 2 multiples of 7 in that range. We see that m must be either $5, 6$, or $0 \mod 7$. 98 is the largest multiple of 7 less than or equal to 100, and no value greater than 98 works. We find our answer to be $98\cdot\frac{3}{7} = \boxed{42}$ □

14. **[6]** In LMTown there are only rainy and sunny days. If it rains one day there's a 30% chance that it will rain the next day. If it's sunny one day there's a 90% chance it will be sunny the next day. Over n days, as n approaches infinity, the percentage of rainy days approaches k%. Find $10k$.

    Proposed by Derek Zhao

    Solution. $\boxed{125}$

    One can set up an equation to solve it. As the percentage approaches a number, we can set it to x. 0.3x+0.1(1-x)=x. Solving, x=1/8, so k=12.5, so the answer is 125. □

15. **[6]** A bag of letters contains 3 L's, 3 M's, and 3 T's. Aidan picks three letters at random from the bag with replacement, and Andrew picks three letters at random from the bag without replacement. Given that the probability that both Aidan and Andrew pick one each of L, M, and T can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers, find $m+n$.

    Proposed by Zachary Perry

    Solution. $\boxed{15}$

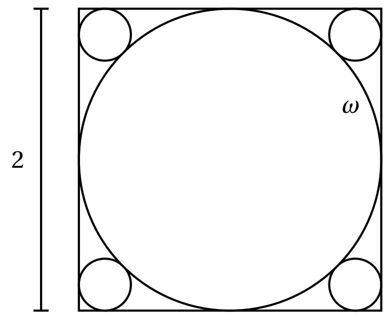
    Aidan's probability is $\frac{3!}{3^3} = \frac{2}{9}$, and Andrew's is $\frac{3!3^3}{9\cdot8\cdot7} = \frac{9}{4\cdot7}$. The product is $\frac{2\cdot9}{9\cdot4\cdot7} = \frac{1}{14}$ □

16. **[6]** Circle ω is inscribed in a square with side length 2. In each corner tangent to 2 of the square's sides and externally tangent to ω is another circle. The radius of each of the smaller 4 circles can be written as $(a-\sqrt{b})$ where a and b are positive integers. Find $a+b$.

    Proposed by Owen Jiang

    Solution. $\boxed{11}$

    The circle internally tangent has radius 1. The diagonal from the center of the cube to the corner is $\sqrt{2}$. Let r be the radius of the smaller circle. Thus, $r + r\sqrt{2} = \sqrt{2}-1$. Solving for r yields $3-\sqrt{8}$. $3+8 = \boxed{11}$. □



17. **[6]** In the nonexistent land of Lexingtopia, there are 10 days in the year, and the Lexingtopian Math Society has 5 members. The probability that no two of the Lexingtopian Math Society's members share the same birthday can be written as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a+b$.

Proposed by BPL

Solution. $\boxed{814}$

Number of ways for different birthdays = $\binom{10}{5}\cdot 5!$. Total number of ways to pick birthdays = 10^5

Probability of different birthdays = $\frac{\binom{10}{5}\cdot 5!}{10^5} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{100000} = \frac{9\cdot 7\cdot 3}{625} = \frac{189}{625}$, so the answer is $189+625=814$ □

18. **[6]** Let $D(n)$ be the number of diagonals in a regular n-gon. Find

$$\sum_{n=3}^{26} D(n).$$

Proposed by Ephram Chun

Solution. $\boxed{2576}$

We know that the formula to find the number of diagonals in an n-sided figure is $\frac{(n)(n-3)}{2}$ Thus

$$\sum_{n=3}^{26} D(n) = \frac{\left(\frac{(26*27*53-84)}{6}\right) - 3(27*13-6)}{2} = \boxed{2576}$$

This □

19. **[6]** Given a square $A_0B_0C_0D_0$ as shown below with side length 1, for all nonnegative integers n, construct points A_{n+1}, B_{n+1}, C_{n+1}, and D_{n+1} on A_nB_n, B_nC_n, C_nD_n, and D_nA_n, respectively, such that

$$\frac{A_nA_{n+1}}{A_{n+1}B_n} = \frac{B_nB_{n+1}}{B_{n+1}C_n} = \frac{C_nC_{n+1}}{C_{n+1}D_n} = \frac{D_nD_{n+1}}{D_{n+1}A_n} = \frac{3}{4}.$$

The sum of the series

$$\sum_{i=0}^{\infty} [A_iB_iC_iD_i] = [A_0B_0C_0D_0] + [A_1B_1C_1D_1] + [A_2B_2C_2D_2]\dots$$

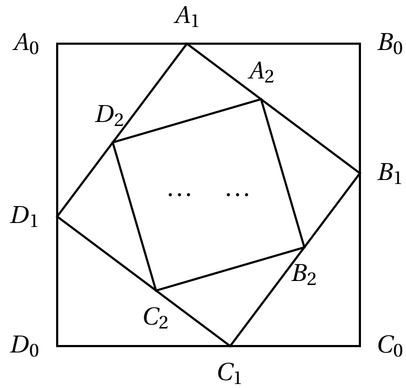
where $[\mathbf{P}]$ denotes the area of polygon $\mathbf{P}$ can be written as $\frac{a}{b}$ where a and b are relatively prime positive integers. Find $a+b$.

Proposed by Ada Tsui

Solution. $\boxed{73}$

In this configuration, consider a square $EFGH$ with a side length x and its successor $E'F'G'H'$. Then

$$EE' : E'F = FF' : F'G = GG' : G'H = HH' : H'E = 3:4,$$



so

$$EE' = FF' = GG' = HH' = \frac{3}{7}x$$

and

$$E'F = F'G = G'H = H'E = \frac{4}{7}x,$$

so

$$E'F' = F'G' = G'H' = H'E' = \frac{5}{7}x$$

because of the right angles in the square.

Then the area of $E'F'G'H'$ is $\frac{5}{7}x \cdot \frac{5}{7}x = \frac{25}{49}x^2$. The area of $EFGH$ is x^2, so

$$\frac{[E'F'G'H']}{[EFGH]} = \frac{\frac{25}{49}x^2}{x^2} = \frac{25}{49}$$

and the area of one square in this configuration is $\frac{25}{49}$ the area of its predecessor.

(Note that if this generalization is not immediately obvious, then experimentation with $ABCD$ to $A'B'C'D'$, $A'B'C'D'$ to $A''B''C''D''$, and so on suggest this generalization.)

Then this is just the geometric series $1 + \frac{25}{49} + \frac{625}{2401} + \cdots$ with the first term 1 and common ratio $\frac{25}{49}$, so its sum is $\frac{1}{1-\frac{25}{49}} = \frac{49}{24}$.

Of course, the answer is $a+b$, and because $\frac{49}{24} = \frac{a}{b}$, $a+b = 49+24 = \boxed{73}$. □

20. **[6]** Let m and n be two real numbers such that

$$\frac{2}{n} + m = 9$$
$$\frac{2}{m} + n = 1$$

Find the sum of all possible values of m plus the sum of all possible values of n.

Proposed by Bang Tam Ngo

Solution. $\boxed{10}$

Multiple both equations so you get

$$\frac{4}{\sqrt{nm}} + \sqrt{nm} + 4 = 9$$

Then solve for $\sqrt{nm}$ like a quadratic equation so that you get

$$(\sqrt{nm} - 4)(\sqrt{nm} - 1) = 0$$

If $\sqrt{nm} = 4$, then

$$n = \frac{4}{9}, m = 36$$

If $\sqrt{nm} = 1$, then

$$n = \frac{1}{9}, m = 9$$

Add them all up together

$$\frac{1}{9} + \frac{4}{9} + 36 + 9 = \frac{410}{9}$$
$$410 + 9 = 419$$

Note: I, Zachary, removed square roots from the original equation, so this solution is mostly correct but outdated. It should be $6 + 3 + \frac{2}{3} + \frac{1}{3} = \boxed{10}$. □

21. **[6]** Let $\sigma(x)$ denote the sum of the positive divisors of x. Find the smallest prime p such that

$$\sigma(p!) \geq 20 \cdot \sigma([p-1]!).$$

Proposed by Aidan Duncan

Solution. $\boxed{19}$

The factors of $p!$ consist of the factors of $(p-1)!$ and the numbers that are p times each of these factors. Thus, the sum of the factors of $p!$ is $p+1$ times the sum of the factors of $(p-1)!$. So, we have $(0.05)(p+1)a \geq a$, $p+1 \geq 20$, so the smallest prime p where this is true is $\boxed{19}$ □

22. **[6]** Let $\triangle ABC$ be an isosceles triangle with $AB = AC$. Let M be the midpoint of side $\overline{AB}$. Suppose there exists a point X on the circle passing through points A, M, and C such that $BMCX$ is a parallelogram and M and X are on opposite sides of line BC. Let segments $\overline{AX}$ and $\overline{BC}$ intersect at a point Y. Given that $BY = 8$, find AY^2.

Proposed by Taiki Aiba

Solution. $\boxed{40}$

Since $\overline{BM} \parallel \overline{XC}$, we have that $\angle MXC = \angle BMX = \theta$. Next, note that $\angle AMX + \angle BMX = 180^\circ$, so

$$\angle AMX = 180^\circ - \angle BMX = 180^\circ - \theta.$$

Then, we have that

$$\angle ACX = 180^\circ - \angle AMX = 180^\circ - (180^\circ - \theta) = \theta.$$

Next, since $\overline{AM} \parallel \overline{XC}$ and $AM = BM = XC$, we have that $AMXC$ is a parallelogram. This means that

$$\angle ACX = \angle AMX \implies \theta = 180^\circ - \theta \implies \theta = 90^\circ.$$

Thus, $AMXC$ is a rectangle, so $\angle BAC = 90^\circ$. Let $XC = x$. Since $AB = AC$, we have that $AB = AC = 2x$. Given that $BY = 8$, by similar triangles, we have that $CY = 4$, which means that $BC = 12$. By the Pythagorean Theorem,

$$(2x)^2 + (2x)^2 = 12^2 \implies 8x^2 = 144 \implies x^2 = 18.$$

Then, by the Pythagorean Theorem again,

$$AX^2 = x^2 + (2x)^2 = 5x^2 = 5 \cdot 18 = 90 \implies AX = 3\sqrt{10}.$$

Finally, by similar triangles, we have that $AY = \frac{2}{3}AX$, so $AY = \frac{2}{3} \cdot 3\sqrt{10} = 2\sqrt{10}$, giving $AY^2 = \boxed{40}$. □

23. **[6]** Kevin chooses 2 integers between 1 and 100, inclusive. Every minute, Corey can choose a set of numbers and Kevin will tell him how many of the 2 chosen integers are in the set. How many minutes does Corey need until he is certain of Kevin's 2 chosen numbers?

Proposed by Kevin Zhao

Solution. $\boxed{13}$

$\boxed{13}$. □

24. **[6]** Evaluate

$$1^{-1}\cdot 2^{-1}+2^{-1}\cdot 3^{-1}+3^{-1}\cdot 4^{-1}+\cdots+(2015)^{-1}\cdot(2016)^{-1} \pmod{2017}.$$

Proposed by Ephram Chun

Solution. $\boxed{2}$

2017 is a prime number, thus it follows that the modular inverses of $1,2,\ldots,p-1$ all exist. I claim that $n^{-1}\cdot(n+1)^{-1}\equiv n^{-1}-(n+1)^{-1} \pmod{p}$ for $n\in\{1,2,\ldots,p-2\}$, with the formula

$$\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$$

Indeed, multiplying both sides of the congruence by $n(n+1)$, we find that

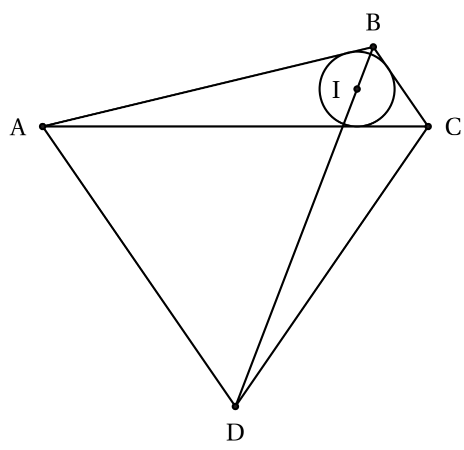
$$1\equiv n(n+1)\cdot(n^{-1}-(n+1)^{-1})\equiv(n+1)-n\equiv 1 \pmod{p},$$

as desired. Thus,

$$1^{-1}\cdot 2^{-1}+2^{-1}\cdot 3^{-1}+3^{-1}\cdot 4^{-1}+\cdots+(p-2)^{-1}\cdot(p-1)^{-1}$$
$$\equiv 1^{-1}-2^{-1}+2^{-1}-3^{-1}+\cdots-(p-1)^{-1} \pmod{p}.$$

This is a telescoping series, which sums to $1^{-1}-(p-1)^{-1}\equiv 1-(-1)^{-1}\equiv\boxed{2} \pmod{p}$, since the modular inverse of -1 is itself. □

25. **[6]** In scalene $\triangle ABC$, construct point D on the opposite side of AC as B such that $\angle ABD=\angle DBC=\angle BCA$ and $AD=DC$. Let I be the incenter of $\triangle ABC$. Given that $BC=64$ and $AD=225$, find BI.



Proposed by Kevin Zhao

Solution. $\boxed{30}$

Note that D is the point on $\triangle ABC$'s circumcenter such that BD bisects $\angle ABC$. We note that since $\angle ACB=\angle CBD=\angle DBA$, then $AB=CD=DA=225$. Ptolemy's tells us that

$$AC\cdot BD=AB\cdot CD+AD\cdot BC=225^2+225\cdot 64=15^2\cdot 17^2$$

so since $AC=BD$ because $AB=CD$, then $AC=BD=15\cdot 17$. Thus, since by Incenter-Excenter we have $DI=DC=DA=15^2$, then $BI=BD-DI=15\cdot 17-15^2=\boxed{30}$. □