13th Annual Spring Lexington Math Tournament - Guts Round - Part 1

Team Name: ______________________________

______________ 1. **[5]** A box contains 1 ball labelled W, 1 ball labelled E, 1 ball labelled L, 1 ball labelled C, 1 ball labelled O, 8 balls labelled M, and 1 last ball labelled E. One ball is randomly drawn from the box. The probability that the ball is labelled E is $\frac{1}{a}$. Find a.

Proposed by Ada Tsui

Solution. $\boxed{7}$

The probability is $\frac{2}{1+1+1+1+1+8+1} = \frac{1}{7}$, so our answer is $\boxed{7}$. □

______________ 2. **[5]** Let

$$\begin{aligned} G+E+N &= 7 \\ G+E+O &= 15 \\ N+T &= 22. \end{aligned}$$

Find the value of $T+O$.

Proposed by Ada Tsui

Solution. $\boxed{30}$

Note that subtracting first two equations gets that $O-N=8$ and so $N+T+O-N=T+O=8+22=\boxed{30}$. □

______________ 3. **[5]** The area of $\triangle LMT$ is 22. Given that $MT=4$ and that there is a right angle at M, find the length of LM.

Proposed by

Solution. $\boxed{11}$

The area of $\triangle LMT$ is $\frac{1}{2}\cdot c\cdot 4 = 22 \to c = \boxed{11}$. □

13th Annual Spring Lexington Math Tournament - Guts Round - Part 2

Team Name: ______________________________

______________ 4. **[5]** Kevin chooses a positive 2-digit integer, then adds 6 times its unit digit and subtracts 3 times its tens digit from itself. Find the greatest common factor of all possible resulting numbers.

Proposed by Kevin Zhao

Solution. $\boxed{7}$

$10a+b+6b-3a=7(a+b)$. If the number is 10, the answer is also $\boxed{7}$ so we are done. □

______________ 5. **[5]** Find the maximum possible number of times circle D can intersect pentagon $GRASS'$ over all possible choices of points G, R, A, S, and S'.

Proposed by Muztaba Syed

Solution. $\boxed{10}$

The circle can intersect each side of the pentagon twice. There are 5 sides, so the answer is $2\cdot 5=\boxed{10}$. We can make an easy construction for this by taking the circumcircle of a regular pentagon and shrinking it slightly. □

______ 6. **[5]** Find the sum of the digits of the integer solution to

$$(\log_2 x) \cdot (\log_4 \sqrt{x}) = 36.$$

Proposed by Kira Tang

Solution. $\boxed{19}$

$\log_4 \sqrt{x} = \frac{1}{2}\log_4 x = \frac{1}{4}\log_2 x$, so $\frac{1}{4}(\log_2 x)^2 = 36$. Solving for x, we get $x = 2^{12} = 4096$. Thus, the sum we are looking for is $4+0+9+6 = \boxed{19}$. □

13th Annual Spring Lexington Math Tournament - Guts Round - Part 3

Team Name: ______________________

______ 7. **[6]** Given that x and y are positive real numbers such that $x^2 + y = 20$, the maximum possible value of $x+y$ can be written as $\frac{a}{b}$ where a and b are relatively prime positive integers. Find $a+b$.

Proposed by Kevin Zhao

Solution. $\boxed{85}$

Note that $y = 20 - x^2$ so $x + y = -x^2 + x + 20$ and so $x = 0.5$ maximizes this. The answer is thus $\frac{81}{4} \to \boxed{85}$. □

______ 8. **[6]** In $\triangle DRK$, $DR = 13$, $DK = 14$, and $RK = 15$. Let E be the point such that $ED = ER = EK$. Find the value of $\lfloor DE + RE + KE \rfloor$

Proposed by Jeff Lin

Solution. $\boxed{24}$

The circumradius is $\frac{65}{8}$ so three times that floored is $\boxed{24}$. □

______ 9. **[6]** Subaru the frog lives on lily pad 1. There is a line of lily pads, numbered 2, 3, 4, 5, 6, and 7. Every minute, Subaru jumps from his current lily pad to a lily pad whose number is either 1 or 2 greater, chosen at random from valid possibilities. There are alligators on lily pads 2 and 5. If Subaru lands on an alligator, he dies and time rewinds back to when he was on lily pad number 1. Find how many times Subaru is expected to die before he reaches pad 7.

Proposed by Corey Zhao

Solution. $\boxed{7}$

Let us say $E(x)$ is the expected number of deaths from lily pad x. Then, $E(7) = E(6) = 0$, $E(5) = E(2) = 1 + E(1)$, $E(4) = \frac{1+E(1)}{2}$, $E(3) = \frac{3+3E(1)}{4}$, so $E(1) = \frac{7+7E(1)}{8} \Rightarrow E(1) = \boxed{7}$. □

13th Annual Spring Lexington Math Tournament - Guts Round - Part 4

Team Name: ______________________________

______________ 10. **[6]** Find the sum of the following series:

$$\sum_{i=1}^{\infty} = \frac{\sum_{j=1}^{i} j}{2^i} = \frac{1}{2^1} + \frac{1+2}{2^2} + \frac{1+2+3}{2^3} + \frac{1+2+3+4}{2^4} + \dots.$$

Proposed by Kevin Zhao

Solution. $\boxed{4}$

Note that

$$\frac{S}{2} = \left(\frac{1}{2^1} + \frac{1+2}{2^2} + \frac{1+2+3}{2^3} + \frac{1+2+3+4}{2^4} + \dots\right) - \left(\frac{1}{2^2} + \frac{1+2}{2^3} + \frac{1+2+3}{2^4} + \dots\right) = S - \frac{S}{2} = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots.$$

Now, subtracting again gets $\frac{S}{4} = \frac{S}{2} - \frac{S}{4} = \frac{1}{2} + \frac{1}{4} + \dots = 1$ meaning that $S = \boxed{4}$. □

______________ 11. **[6]** Let $\varphi(x)$ be the number of positive integers less than or equal to x that are relatively prime to x. Find the sum of all x such that $\varphi(\varphi(x)) = x-3$. Note that 1 is relatively prime to every positive integer.

Proposed by Ephram Chun

Solution. $\boxed{9}$

The equation for $\phi(x)$, $x = p_1^{q_1} \cdot p_2^{q_2} \cdot \dots \cdot p_n^{q_n}$, is

$$\phi(x) = x \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdot \dots \cdot \left(1 - \frac{1}{p_n}\right)$$

.

We know that for $x > 1$, $\phi(x) \le x - 1$, with equality if and only if x is prime. We can achieve $\phi(\phi(x)) = x-3$ in three ways: $\phi(x) = x-1, \phi(x-1) = x-3$ or $\phi(x) = x-2, \phi(x-2) = x-3$, and $\phi(x) = x-3 = 1, \phi(1) = 1$. We can easily check that the third case doesn't work.

Case 1: $\phi(x) = x-1$ means that x is prime. Experimenting leads to the fact that $\phi(x-1) = x-3$ only occurs for $x-1 = 4$. This leads to $x = 5$, which satisfies the requirement that x is prime. So, we have attained one solution.

Case 2: $\phi(x) = x-2$ only occurs for $x-2 = 4$. We continue and find that $\phi(\phi(4)) = 1$, which satisfies our requirement, so that $x = 4$ is another solution.

In total, we have $5+4 = \boxed{9}$ □

______________ 12. **[6]** On a piece of paper, Kevin draws a circle. Then, he draws two perpendicular lines. Finally, he draws two perpendicular rays originating from the same point (an L shape). What is the maximum number of sections into which the lines and rays can split the circle?

Proposed by Kevin Zhao, modified by Jeff Lin

Solution. $\boxed{9}$

Draw the L to intersect the circle 4 times, and the perpendicular lines such that one line intersects the L inside the circle twice and the other intersects it once. This results in $\boxed{9}$ regions. □

13th Annual Spring Lexington Math Tournament - Guts Round - Part 5

Team Name: ______________________________

________ 13. **[7]** In quadrilateral $ABCD$, $\angle A = 90°$, $\angle C = 60°$, $\angle ABD = 25°$, and $\angle BDC = 5°$. Given that $AB = 4\sqrt{3}$, the area of quadrilateral $ABCD$ can be written as $a\sqrt{b}$. Find $10a + b$.

Proposed by Ephram Chun

Solution. $\boxed{83}$

I claim that if we reflect point C across the perpendicular bisector of line segment $\overline{BD}$ to get point C', then we get a right triangle ABC', where point D is on side AC'. We see that this happens because $\angle ABC' = \angle BDC' + \angle ABD = 5° + 25° = 30°$, $\angle BC'D = \angle BC'A = 60°$, and $\angle BAC' = 90°$. We also know that $\angle ADC' = \angle BDA + \angle BDC' = (180° - 90° - 25°) + (180° - 60° - 5°) = 65° + 115° = 180°$, so point D is on side AC'. By extension, we now know that right triangle ABC' is a $30-60-90$ right triangle, where $\angle A = 90°$, $\angle B = 30°$, and $\angle C' = 60°$.

We know that right triangle ABC' has the same area as quadrilateral $ABCD$ because triangles BCD and $BC'D$ have the same areas (this reflection preserves areas), and triangle ABD is unchanged. Since we are given that $AB = 4\sqrt{3}$, it follows that the other leg, AC', has length 4. We have that the area of right triangle ABC', and thereby the area of quadrilateral $ABCD$, is $\frac{1}{2} \cdot 4 \cdot 4\sqrt{3} = 8\sqrt{3}$. Thus our answer is $\boxed{83}$. □

________ 14. **[7]** The value of

$$\sum_{n=2}^{6} \left(\frac{n^4+1}{n^4-1}\right) - 2\sum_{n=2}^{6}\left(\frac{n^3-n^2+n}{n^4-1}\right)$$

can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $100m + n$.

Proposed by Ephram Chun

Solution. $\boxed{19770}$

$$\sum_{n=2}^{6} \frac{n^4+1}{n^4-1} - 2\sum_{n=2}^{6}\frac{n^3-n^2+n}{n^4-1} = \sum_{n=1}^{6} \frac{n^4-2n^3+2n^2-2n+1}{n^4-1} = \sum_{n=1}^{6} \frac{(n-1)(n^3-n^2+n-1)}{(n+1)(n^3-n^2+n-1)} = \sum_{n=1}^{6} \frac{n-1}{n+1}$$

$$\frac{1}{3} + \frac{2}{4} + \cdots + \frac{5}{7} = \frac{197}{70}$$

$$19700 + 70 = \boxed{19770}$$

□

________ 15. **[7]** Positive real numbers x and y satisfy the following 2 equations.

$$x^{1+x^{1+x^{1+\dots}}} = 8$$

$$\sqrt[24]{y + \sqrt[24]{y + \sqrt[24]{y + \dots}}} = x$$

Find the value of $\lfloor y \rfloor$.

Proposed by Kevin Zhao

Solution. $\boxed{254}$

Note that we can substitute and get that $x^{1+8} = 8 \to x = \sqrt[3]{2}$. Hence, doing the same thing with the second equation gets $\sqrt[24]{y+x} = x = \sqrt[3]{2}$ so taking both sides to the 24the power gets that $y + x = 256$ meaning that $\lfloor y \rfloor = 256 - \lceil \sqrt[3]{2} \rceil = \boxed{254}$. □

13th Annual Spring Lexington Math Tournament - Guts Round - Part 6

Team Name: ________________________

16. **[7]** Given that x and y are positive real numbers such that $x^3 + y = 20$, the maximum possible value of $x + y$ can be written as $\frac{a\sqrt{b}}{c} + d$ where a, b, c, and d are positive integers such that $\gcd(a, c) = 1$ and b is square-free. Find $a + b + c + d$.

Proposed by Kevin Zhao

Solution. $\boxed{34}$

Note that $y = 20 - x^2$ so $x + y = -x^3 + x + 20$ and so the derivative is $-3x^2 + 1$ meaning $x = \frac{1}{\sqrt{3}}$ maximizes the value. The answer is thus $20 + \frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{9} = 20 + \frac{2\sqrt{3}}{9}$ meaning out answer is $20+2+3+9 = \boxed{34}$. □

17. **[7]** In $\triangle DRK$, $DR = 13$, $DK = 14$, and $RK = 15$. Let E be the intersection of the altitudes of $\triangle DRK$. Find the value of $\lfloor DE + RE + KE \rfloor$

Proposed by Jeff Lin

Solution. $\boxed{24}$

□

18. **[7]** Subaru the frog lives on lily pad 1. There is a line of lily pads, numbered 2, 3, 4, 5, 6, and 7. Every minute, Subaru jumps from his current lily pad to a lily pad whose number is either 1 or 2 greater, chosen at random from valid possibilities. There are alligators on lily pads 2 and 5. If Subaru lands on an alligator, he dies and time rewinds back to when he was on lily pad number 1. Find the expected number of jumps it takes Subaru to reach pad 7.

Proposed by Corey Zhao, modified by Jeff Lin

Solution. $\boxed{15}$

Let $E(x)$ be the expected number of jumps for Subaru to get to pad 6 from pad x. Then, $E(6) = 0$, $E(4) = 1 + \frac{1}{2}E(1).E(3) = 1 + \frac{1}{2}E(4) + \frac{1}{2}E(1). = \frac{3E(1)+6}{4}.E(1) = 1 + \frac{1}{2}E(1) + \frac{1}{2}E(3)$. Solving, we get $E(1) = 14$. After getting to pad 6, we need one more move. This gives us $\boxed{15}$. □

13th Annual Spring Lexington Math Tournament - Guts Round - Part 7

Team Name: ______________________________

This set has problems whose answers depend on one another.

______________ 19. **[8]** Let B be the answer to Problem 20 and let C be the answer to Problem 21. Given that $f(x) = x^3 - Bx - C = (x-r)(x-s)(x-t)$ where r, s, and t are complex numbers, find the value of $r^2 + s^2 + t^2$.

Proposed by Kevin Zhao

Solution. $\boxed{52}$

Note that $x^3 = Bx + C$ for all such roots so $x^2 = B + \frac{C}{x}$ and Vieta's shows us that

$$\frac{C}{r} + \frac{C}{s} + \frac{C}{t} = C\left(\frac{1}{r} + \frac{1}{s} + \frac{1}{t}\right) = C \cdot \frac{-B}{C} = -B$$

so the sum is $\boxed{2B}$. Later, we see that $2B = 2 \cdot 26 = \boxed{52}$. □

______________ 20. **[8]** Let A be the answer to Problem 19 and let C be the answer to Problem 21. Circles ω_1 and ω_2 meet at points X and Y. Let point $P \neq Y$ be the point on ω_1 such that PY is tangent to ω_2, and let point $Q \neq Y$ be the point on ω_2 such that QY is tangent to ω_1. Given that $PX = A$ and $QX = C$, find XY.

Proposed by Kevin Zhao

Solution. $\boxed{26}$

Since $\angle PYX = \angle YQX$ and $\angle QYX = \angle YPX$, then $\triangle PXY \sim \triangle YXQ$ and $XY^2 = PX \cdot QX = AC$. Since $A = 2B$, then $B = 2C$ and so later because $C = 13$, $B = \boxed{26}$. □

______________ 21. **[8]** Let A be the answer to Problem 19 and let B be the answer to Problem 20. Given that the positive difference between the number of positive integer factors of A^B and the number of positive integer factors of B^A is D, and given that the answer to this problem is an odd prime, find $\frac{D}{B} - 40$.

Proposed by Kevin Zhao

Solution. $\boxed{13}$

Note that the numbers are $(2C)^{4C}$ and $(4C)^{2C}$ which are equivalent to $2^{4C} \cdot C^{4C}$ and $2^{4C} \cdot C^{2C}$. This means that the answer to the problem is going to be $\frac{(4C+1)(2C)}{2C} - 40 = 4C - 39 = C \rightarrow C = \boxed{13}$. □

13th Annual Spring Lexington Math Tournament - Guts Round - Part 8

Team Name: ______________________________

______________ 22. **[8]** Let $v_p(n)$ for a prime p and positive integer n output the greatest nonnegative integer x such that p^x divides n. Find

$$\sum_{i=1}^{50} \sum_{p=1}^{i} \binom{v_p(i)+1}{2},$$

where the inner summation only sums over primes p between 1 and i.

Proposed by Jeff

Solution. $\boxed{155}$

switch summation □

23. **[8]** Let a, b, and c be positive real solutions to the following equations.

$$\frac{2b^2+2c^2-a^2}{4}=25$$
$$\frac{2c^2+2a^2-b^2}{4}=49$$
$$\frac{2a^2+2b^2-c^2}{4}=64$$

The area of a triangle with side lengths a, b, and c can be written as $\frac{x\sqrt{y}}{z}$ where x and z are relatively prime positive integers and y is square-free. Find $x+y+z$.

Proposed by Kevin Zhao

Solution. $\boxed{46}$

Note that a, b, and c are sides of a triangle with medians 5, 7, and 8 from the median formula. Now, this means that since the area of the triangle formed by lengths a, b, and c is $\frac{4}{3}$ of that formed by its medians, then we need the area of the triangle with side lengths 5, 7, and 8. The 60 degree angle in this triangle shows the area as $\frac{1}{2}\cdot 5\cdot 8\cdot \frac{\sqrt{3}}{2}=10\sqrt{3}$, so our answer is $10\sqrt{3}\cdot\frac{4}{3}=\frac{40\sqrt{3}}{3}\rightarrow\boxed{46}$. □

24. **[8]** Alan, Jiji, Ina, Ryan, and Gavin want to meet up. However, none of them know when to go, so they each pick a random 1 hour period from 5 AM to 11 AM to meet up at Alan's house. Find the probability that there exists a time when all of them are at the house at one time.

Proposed by Ephram Chun

Solution. $\boxed{2599}$

This is a classic geometric probability problem in 2 dimensions, and we see that it can be generalized into more dimensions. For 2-D, drawing the graph is relatively simple and it is easy to see that the area where they all meet each other is 2 parallelograms plus a square. If we let each hour be 1 unit, each parallelogram has base 1 and height 5, and the square is a unit square, giving us $5+5+1=11$ for total area. The entire area is 36, so the answer is $\frac{11}{36}$ for 2-D.

Visualizing the 3-D case takes more time and effort, but we can eventually see that we have 3 tilted rectangular prisms, which have base 1 and height 5. This gives us a total volume of $5*3+1=16$, but the entire cube has volume 6^3, so the answer is $\frac{16}{216}$.

Now, it is pretty easy to convince yourself that the answer is always $\frac{[5n+1]}{[6^n]}$, so in 5 dimensions the answer is $\frac{21}{6^5}$, or $\frac{7}{2592}$. Thus our answer is $\boxed{2599}$ □

13th Annual Spring Lexington Math Tournament - Guts Round - Part 9

Team Name: ______________________________

______________ 25. **[8]** Let n be the number of registered participants in this LMT. Estimate the number of digits of $\left[\binom{n}{2}\right]!$ in base 10. If your answer is A and the correct answer is C, then your score will be

$$\left\lfloor \max\left(0, 20 - \left|\ln\left(\frac{A}{C}\right)\cdot 5\right|\right)\right\rfloor.$$

Proposed by

Solution. $\boxed{11744}$

□

______________ 26. **[8]** Let γ be the minimum value of x^x over all real numbers x. Estimate $\lfloor 10000\gamma \rfloor$. If your answer is A and the correct answer is C, then your score will be

$$\left\lfloor \max\left(0, 20 - \left|\ln\left(\frac{A}{C}\right)\cdot 5\right|\right)\right\rfloor.$$

Proposed by Ephram Chun

Solution. $\boxed{3678}$

□

______________ 27. **[8]** Let

$$E = \log_{13} 1 + \log_{13} 2 + \log_{13} 3 + \cdots + \log_{13} 513513.$$

Estimate $\lfloor E \rfloor$. If your answer is A and the correct answer is C, your score will be

$$\left\lfloor \max\left(0, 20 - \left|\ln\left(\frac{A}{C}\right)\cdot 5\right|\right)\right\rfloor.$$

Proposed by Ephram Chun

Solution. $\boxed{2432286}$

$$E = \log_{13} 513513!$$

$$\lfloor E \rfloor = \boxed{2432286}$$

□