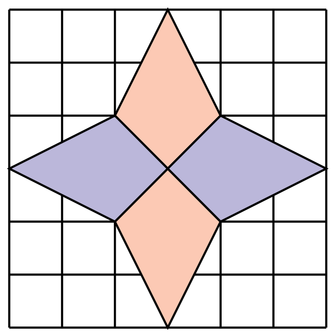
Accuracy Round Solutions

Lexington High School

May 14th, 2022

1. **[6]** Kevin colors a ninja star on a piece of graph paper where each small square has area 1 square inch. Find the area of the region colored, in square inches.



Proposed by Kevin Zhao

Solution. $\boxed{12}$

Simple geometry figures gives us $\boxed{12}$ □

2. **[8]** Let $a \spadesuit b = \frac{a^2-b^2}{2b-2a}$. Given that $3 \spadesuit x = -10$, compute x.

Proposed by Muztaba Syed

Solution. $\boxed{17}$

To make computation easier simplify $a@b = \frac{a^2-b^2}{2b-2a} = \frac{(a+b)(a-b)}{-2(a-b)} = \frac{a+b}{-2}$ with difference of squares ($a-b$ is non-0 for the purposes of our computation). Now we have $3+x = -10 \cdot -2 \implies x = \boxed{17}$ □

3. **[10]** Find the difference between the greatest and least values of $\text{lcm}(a,b,c)$, where a, b, and c are distinct positive integers between 1 and 10, inclusive.

Proposed by Ephram Chun

Solution. $\boxed{626}$

The smallest is $\text{lcm}(1,2,4) = 4$, and the greatest is $\text{lcm}(7,9,10) = 630$, so the answer is $\boxed{626}$. □

4. **[12]** Kevin runs uphill at a speed that is 4 meters per second slower than his speed when he runs downhill. Kevin takes a total of 80 seconds to run up and down a hill on one path. Given that the path is 300 meters long (he travels 600 meters total), find how long Kevin takes to run up the hill in seconds.

Proposed by Kevin Zhao

Solution. $\boxed{50}$

We note that letting D be the distance, we have $\frac{D}{x} + \frac{D}{x+4} = 80$ where we want $\frac{D}{x}$. Solving for x gets $x = 6$ and so our answer is $\frac{300}{6} = \boxed{50}$. □

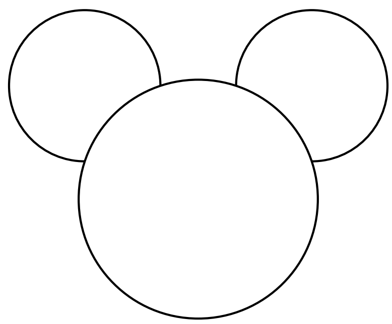
5. **[14]** A bag contains 5 identical blue marbles and 5 identical green marbles. In how many ways can 5 marbles from the bag be arranged in a row if each blue marble must be adjacent to at least 1 green marble?

   Proposed by Ephram Chun

   Solution. $\boxed{16}$

   We proceed with casework. If there is 1 blue marble and 4 green marbles then there are $\binom{5}{1} = 5$ ways to arrange the marbles. If there are 2 blue marbles and 3 green marbles then there is a total of $\frac{5!}{3!2!} = 10$ total ways to arrange GBGBG. But 2 arrangements do not satisfy the conditions which are $GGGBB$ and $BBGGG$. Therefore there are 8 possible arrangements from this case. If there are 3 blue marbles then it must be BGBGB, GBBGB, BGBBG in order for each blue marble to be adjacent to at least 1 green marble. Thus, our answer is $5+8+3 = \boxed{16}$ □

6. **[16]** Jacob likes to watch Mickey Mouse Clubhouse! One day, he decides to create his own Mickey Mouse head shown below, with two circles ω_1 and ω_2 and a circle ω, and centers O_1, O_2, and O, respectively. Let ω_1 and ω meet at points P_1 and Q_1, and let ω_2 and ω meet at points P_2 and Q_2. Point P_1 is closer to O_2 than Q_1, and point P_2 is closer to O_1 than Q_2. Given that P_1 and P_2 lie on O_1O_2 such that $O_1P_1 = P_1P_2 = P_2O_2 = 2$, and $Q_1O_1 \parallel Q_2O_2$, the area of ω can be written as $n\pi$. Find n.



   Proposed by Kevin Zhao

   Solution. $\boxed{10}$

   Note that $\angle P_1O_1Q_1 = \angle P_2O_2Q_2 = 90°$ because of the parallelity and symmetry. Now, we notice that letting M be the midpoint of O_1O_2, then because $\angle OO_1P_1 = \angle OO_1Q_1 = 45°$, then because $OM||O_1O_2$, we have that both $\triangle O_1MO$ and $\triangle O_2MO$ are isosceles right triangles so $OM = 3$. Thus, ω's area can be expressed as $\pi \cdot r^2$ where we want r^2; We note that $r^2 = OM^2 + MP_1^2 = 3^2 + 1^2 = \boxed{10}$. □

7. **[18]** A teacher wishes to separate her 12 students into groups. Yesterday, the teacher put the students into 4 groups of 3. Today, the teacher decides to put the students into 4 groups of 3 again. However, she doesn't want any pair of students to be in the same group on both days. Find how many ways she could form the groups today.

   Proposed by Ephram Chun

   Solution. $\boxed{1296}$

   We let the groups of students on the previous day be A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3, and D_1, D_2, D_3. We know that A_1, A_2, and A_3 must be in different groups. The remaining group must contain one student from group B, one from C, and one from D. There are $3\cdot3\cdot3 = 27$ ways to do this. The remaining six students can be put into groups with the students from group A in $6\cdot2\cdot2\cdot2 = 48$ ways. Therefore there are $27\cdot48 = \boxed{1296}$ ways. □

8. **[20]** A ray originating at point P intersects a circle with center O at points A and B, with $PB > PA$. Segment $\overline{OP}$ intersects the circle at point C. Given that $PA = 31$, $PC = 17$, and $\angle PBO = 60°$, find the radius of the circle.

   Proposed by Ephram Chun

   Solution. $\boxed{224}$

   Let r be the radius of the circle. By Power of a Point, we have that $PA(PB) = PC(PC+2r)$. We are given that $PA = 31$ and $PC = 17$, so we have that $31PB = 17^2 + 34r$. Next, to find PB, we can use the fact that $\angle PBO = 60°$. We have that

OA and OB are both radii of the circle, so $OA = OB$. Since $\angle PBO = 60^\circ$ and $OA = OB$, it follows that $\angle BAO = 60^\circ$, so $\angle AOB = 60^\circ$. We have that triangle AOB is equilateral, so we get that $AB = r$, and $PB = PA + AB = 31 + r$. Substituting, we get that $31(31 + r) = 17^2 + 34r$. Simplifying, we get that $961 + 31r = 289 + 34r$, so $3r = 672$. Finally, we get that $r = 224$, so the radius of the circle is $\boxed{224}$ □

9. **[22]** A rook is randomly placed on an otherwise empty 8×8 chessboard. Owen makes moves with the rook by randomly choosing 1 of the 14 possible moves. Find the expected value of the number of moves it takes Owen to move the rook to the top left square. Note that a rook can move any number of squares either in the horizontal or vertical direction each move.

   Proposed by Owen Jiang

   Solution. $\boxed{70}$

   States also works □

10. **[24]** In a room, there are 100 light switches, labeled with the positive integers $\{1, 2, \dots, 100\}$. They're all initially turned off. On the ith day for $1 \le i \le 100$, Bob flips every light switch with label number k divisible by i a total of $\frac{k}{i}$ times. Find the sum of the labels of the light switches that are turned on at the end of the 100th day.

    Proposed by BPL

    Solution. $\boxed{584}$

    Let us consider the kth locker. In order for it to be open at the end, it must have been flipped an odd number of times. Additionally, if $ab = k$, then the ath student will have flipped k b times, and the bth student will have flipped it a times, flipping it a total of $a + b$ times. This applies to any factor pair of k, meaning that the number of times k is flipped is just the sum of all of its factors.

    That means all open lockers must have the sum of their factors be odd. Let us consider some prime p that is in the prime factorization of k. Using the formula for the sum of factors of a number, if that prime appears a times in k's prime factorization, it will multiply the sum of factors by $(1 + p + p^2 + \dots p^a)$. So, we need that value to be odd, which means we either need a to be even, or p to be 2. Using this, we can systematically list all lockers that will be left open, and sum their numbers up. 1, 2, 4, 8, 16, 32, 64 $\Rightarrow$ 127 9, 18, 36, 72 $\Rightarrow$ 135 25, 50, 100 $\Rightarrow$ 175 49, 98 $\Rightarrow$ 147

    So, the sum of all lockers that will be left open is $127 + 135 + 175 + 147 = \boxed{584}$ □

11. **[TIEBREAKER]** Let L be the number of times the letter L appeared on the Speed Round, M be the number of times the letter M appeared on the Speed Round, and T be the number of times the letter T appeared on the Speed Round. Find the value of LMT.

    Proposed by Kevin Zhao

    Solution. $\boxed{}$

    $LMT = 172 \cdot 89 \cdot 368 = \boxed{5633344}$. □