Team Round

Lexington High School

May 14th, 2022

1. **[7]** Derek and Jacob have a cake in the shape a rectangle with dimensions 14 inches by 9 inches. They make a deal to split it: Derek takes home the portion of the cake that is less than one inch from the border, while Jacob takes home the remainder of the cake. Let $D : J$ be the ratio of the amount of cake Derek took to the amount of cake Jacob took, where D and J are relatively prime positive integers. Find $D + J$.

2. **[7]** Five people are standing in a straight line, and the distance between any two people is a unique positive integer number of units. Find the least possible distance between the leftmost and rightmost people in the line in units.

3. **[7]** Let the four real solutions to the equation $x^2 + \frac{144}{x^2} = 25$ be r_1, r_2, r_3, and r_4. Find $|r_1| + |r_2| + |r_3| + |r_4|$.

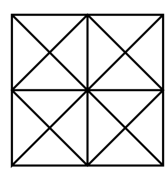
4. **[7]** Jeff has a deck of 12 cards: 4 Ls, 4 Ms, and 4 Ts. Armaan randomly draws three cards without replacement. The probability that he takes 3 Ls can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

5. **[7]** Find the sum

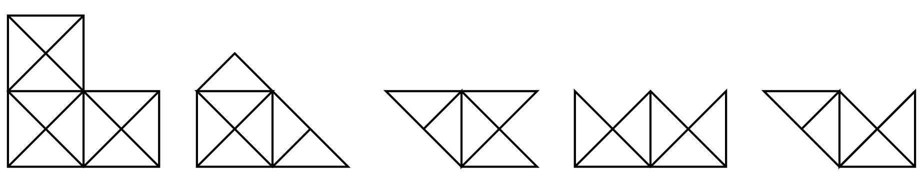
$$\sum_{n=1}^{2020} \gcd(n^3 - 2n^2 + 2021, n^2 - 3n + 3).$$

6. **[7]** For all y, define cubic $f_y(x)$ such that $f_y(0) = y$, $f_y(1) = y + 12$, $f_y(2) = 3y^2$, $f_y(3) = 2y + 4$. For all y, $f_y(4)$ can be expressed in the form $ay^2 + by + c$ where a, b, c are integers. Find $a + b + c$.

7. **[7]** Kevin has a square piece of paper with creases drawn to split the paper in half in both directions, and then each of the four small formed squares diagonal creases drawn, as shown below.



Find the sum of the corresponding numerical values of figures below that Kevin can create by folding the above piece of paper along the creases. (The figures are to scale.) Kevin cannot cut the paper or rip it in any way.



1 2 4 8 16

8. **[7]** The 53-digit number

$$37,984,318,966,591,152,105,649,545,470,741,788,308,402,068,827,142,719$$

can be expressed as n^{21} where n is a positive integer. Find n.

9. **[7]** Let r_1, r_2, ..., r_{2021} be the not necessarily real and not necessarily distinct roots of $x^{2022} + 2021x = 2022$. Let $S_i = r_i^{2021} + 2022r_i$ for all $1 \le i \le 2021$. Find $\left|\sum_{i=1}^{2021} S_i\right| = |S_1 + S_2 + \ldots + S_{2021}|$.

10. **[7]** In a country with 5 distinct cities, there may or may not be a road between each pair of cities. It's possible to get from any city to any other city through a series of roads, but there is no set of three cities $\{A, B, C\}$ such that there are roads between A and B, B and C, and C and A. How many road systems between the five cities are possible?

Team Long Answer

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This test is a team proof round. Each team has 90 minutes for both this round and the team computational round, and takes them at the same time. The total number of obtainable points in this team proof round is 130, and the team computational round has 70 obtainable points.

Each problem submitted in a new subsection should be on a separate page labeled with the problem number and the team name. Note that problems in the same subsection don't need to be on different pages.

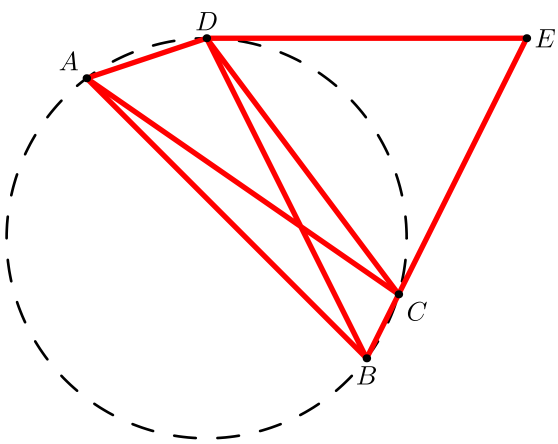
In each problem, you may use any results from earlier in the round as if they were proven, even if your team couldn't solve them. All sub-problems of each problem use the same diagram and same labeled points.

Each problem's point value is bracketed in bold before the question. For example, [**4**] means a problem is worth 4 points. Every section and subsection's total point value is also bracketed in their corresponding titles.

Finally, credits to **Evan Chen** for the LaTeX package.

§1 Some useful properties

Definition 1.1. A set of points is **concyclic** if there is a circle such that all points in this set lie on the circle. A quadrilateral or other polygon is **cyclic** if the set of its vertices is concyclic.



Fact 1.2. $\angle ADB = \angle ACB$ if and only if quadrilateral $ABCD$ is cyclic.

Fact 1.3. $\angle ABC + \angle CDA = 180°$ if and only if quadrilateral $ABCD$ is cyclic.

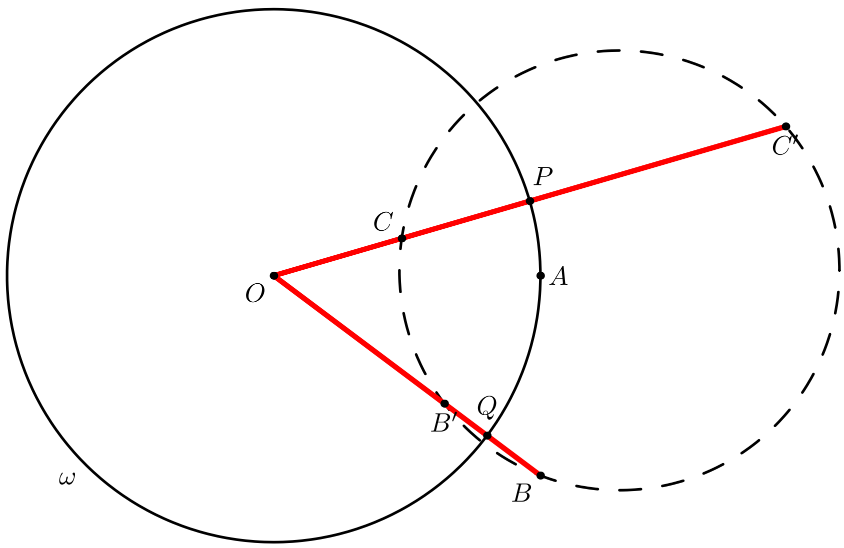
Fact 1.4. $\angle ECD = \angle BAD$ if and only if quadrilateral $ABCD$ is cyclic. (This comes directly from the previous fact, since $\angle ECD + \angle DCB = \angle BAD + \angle DCB = 180°$.)

Theorem 1.5 (Power of a Point)

Circle ω with center O has radius r. A line l intersects ω at two points A and B. Let P be another point on l. Then, $PA \cdot PB = PO^2 - r^2$.

§2 Introduction to Inversion [55]

In the Euclidean Plane, let O be the center of a circle ω with radius $r > 0$. An **inversion** upon ω transforms each point P in the plane to a point P', such that $OP \cdot OP' = r^2$ and both ray $\overrightarrow{OP}$ and ray $\overrightarrow{OP'}$ are pointing in the same direction. Inversions have many interesting properties. Let's learn!



Let A be a point on ω and let B and C be points distinct from O such that neither B nor C is on the circle.

Fact 2.1. After an inversion upon ω, A is sent to itself.

Fact 2.2. After an inversion upon ω sends B to B' and C to C', the 4 points B, C, B', and C' are concyclic.

Fact 2.3. Segment OA is tangent to the circumcircle of $\triangle ABB'$.

Fact 2.4. Triangles OBC and $OC'B'$ are similar.

§2.1 Properties of inverting self-invertable circles [25]

Let circle γ with positive radius have center O_1. It is given that γ's inversion upon another circle ω, with center O, transforms γ to itself, but only a finite number of points on γ are sent to themselves via the inversion upon ω. The intersection of γ and ω consists of two points denoted as T_1 and T_2.

Problem 2.5. [**5**] Prove that there exists a circle Ω that is sent to itself by an inversion upon a different circle Γ. *Hint: Use Power of a Point.*

Problem 2.6. [**3**] Let A be a point on γ. Prove that inverting upon ω sends A to the intersection of γ and AO, distinct from A.

Problem 2.7. [**3**] Prove that both OT_1 and OT_2 are tangent to γ. *Hint: Use Power of a Point.*

Problem 2.8. [**4**] Prove that both O_1T_1 and O_1T_2 are tangent to ω.

Problem 2.9. [**5**] Prove that the 4 points O, T_1, O_1, and T_2 are concyclic.

Problem 2.10. [**5**] Prove that O_1 is sent to a point O_1' on T_1T_2 when inverted upon ω.

§2.2 Inverting circles passing through center of inversion [30]

It seems that when inverted upon ω, a circle passing through O becomes a line. Let's prove it! Using the previous section's definitions, let point D be on line T_1T_2 and D' be the intersection of OD and the circumcircle of $\triangle OT_1T_2$. (Remember that this circumcircle also passes through O_1.) Let X be the intersection of OO_1 and T_1T_2.

Problem 2.11. [**4**] Prove that $O_1D' \perp OD$.

Problem 2.12. [**5**] Prove that the four points X, D, O_1, and D' are concyclic.

Problem 2.13. [**7**] Prove that $OD \cdot OD' = r^2$.

Problem 2.14. [**9**] Prove that the inverse of any line (not passing through O) about ω is a circle passing through O.

Problem 2.15. [**5**] Prove that the inverse of any circle passing through O about ω is a line. *Hint: What happens if you invert about a circle twice?*

§3 A Radical Axis Relation [45]

In acute $\triangle ABC$, let D be the foot of the altitude from A to BC. Let E be the foot of the altitude from D to AB and let F be the foot of the altitude from D to AC. Let ω denote the circle with diameter AD and let Ω denote the circumcircle of $\triangle ABC$. Denote $G \neq A$ as the second intersection of ω and Ω, and denote γ as the circle with center A and radius AD.

Problem 3.1. [**3**] Prove that E and F lie on ω.

Problem 3.2. [**3**] Find the points D, E, and F are sent to when inverted upon γ.

Problem 3.3. [**4**] Find the result when ω is inverted upon γ. *Hint: Three points uniquely define a circle passing through them.*

Problem 3.4. [**3**] Prove that when inverted upon γ, point G is sent to a point G' on line BC.

Problem 3.5. [**4**] Prove that B, E, F, and C are concyclic.

Problem 3.6. [**5**] Find the result when line EF is inverted upon γ.

Problem 3.7. [**12**] Let X and Y be where γ intersects Ω. Prove that both X and Y lie on line EF.

Problem 3.8. [**5**] Prove that G' also lies on EF.

Problem 3.9. [**5**] Prove that the four points G, G', B, and E are concyclic.

Problem 3.10. [**1**] Prove that the four points G, G', C, and F are concyclic.

§4 A Challenge Problem [30]

Problem 4.1. [**30**] Obtuse $\triangle ABC$ with $\angle ABC > 90^\circ$ has circumcenter O. Let AC and BO intersect at D, and let E be a point on OD such that $OB^2 = OD \cdot OE$. Prove that B is the incenter of $\triangle ACE$.