Speed Round

Lexington High School

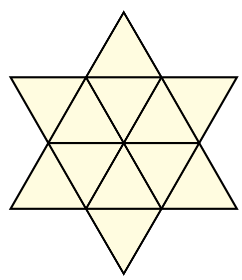
May 14th, 2022

1. **[6]** Aidan walks into a skyscraper's first floor lobby and takes the elevator up 50 floors. After exiting the elevator, he takes the stairs up another 10 floors, then takes the elevator down 30 floors. Find the floor number Aidan is currently on.

2. **[6]** Jeff flips a fair coin twice and Kaylee rolls a standard 6-sided die. The probability that Jeff flips 2 heads and Kaylee rolls a 4 is P. Find $\frac{1}{P}$.

3. **[6]** Given that $a \odot b = a + \frac{a}{b}$, find $(4 \odot 2) \odot 3$.

4. **[6]** The following star is created by gluing together twelve equilateral triangles each of side length 3. Find the outer perimeter of the star.



5. **[6]** In Lexington High School's Math Team, there are 40 students: 20 of whom do science bowl and 22 of whom who do LexMACS. What is the least possible number of students who do both science bowl and LexMACS?

6. **[6]** What is the least positive integer multiple of 3 whose digits consist of only 0s and 1s? The number does not need to have both digits.

7. **[6]** Two fair 6-sided dice are rolled. The probability that the product of the numbers rolled is at least 30 can be written as $\frac{a}{b}$ where a and b are relatively prime positive integers. Find $a + b$.

8. **[6]** At the LHS Math Team Store, 5 hoodies and 1 jacket cost \$13, and 5 jackets and 1 hoodie cost \$17. Find how much 15 jackets and 15 hoodies cost, in dollars.

9. **[6]** Eric wants to eat ice cream. He can choose between 3 options of spherical ice cream scoops. The first option consists of 4 scoops each with a radius of 3 inches, which costs a total of \$3. The second option consists of a scoop with radius 4 inches, which costs a total of \$2. The third option consists of 5 scoops each with diameter 2 inches, which costs a total of \$1. The greatest possible ratio of volume to cost of ice cream Eric can buy is $n\pi$ cubic inches per dollar. Find $3n$.

10. **[6]** Jen claims that she has lived during at least part of each of five decades. What is the least possible age that Jen could be? (Assume that age is always rounded down to the nearest integer.)

11. **[6]** A positive integer n is called *Maisylike* if and only if n has fewer factors than $n-1$. Find the sum of the values of n that are *Maisylike*, between 2 and 10, inclusive.

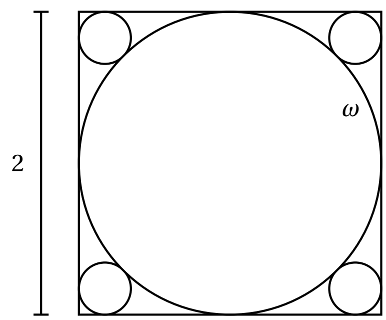
12. **[6]** When Ginny goes to the nearby boba shop, there is a 30% chance that the employee gets her drink order wrong. If the drink she receives is not the one she ordered, there is a 60% chance that she will drink it anyways. Given that Ginny drank a milk tea today, the probability she ordered it can be written as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find the value of $a + b$.

13. **[6]** Alex selects an integer m between 1 and 100, inclusive. He notices there are the same number of multiples of 5 as multiples of 7 between m and $m+9$, inclusive. Find how many numbers Alex could have picked.

14. **[6]** In LMTown there are only rainy and sunny days. If it rains one day there's a 30% chance that it will rain the next day. If it's sunny one day there's a 90% chance it will be sunny the next day. Over n days, as n approaches infinity, the percentage of rainy days approaches k%. Find $10k$.

15. **[6]** A bag of letters contains 3 L's, 3 M's, and 3 T's. Aidan picks three letters at random from the bag with replacement, and Andrew picks three letters at random from the bag without replacement. Given that the probability that both Aidan and Andrew pick one each of L, M, and T can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers, find $m+n$.

16. **[6]** Circle ω is inscribed in a square with side length 2. In each corner tangent to 2 of the square's sides and externally tangent to ω is another circle. The radius of each of the smaller 4 circles can be written as $(a-\sqrt{b})$ where a and b are positive integers. Find $a+b$.



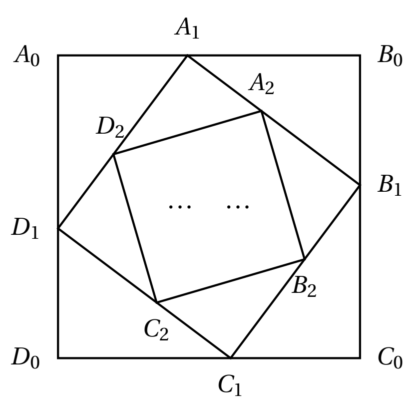
17. **[6]** In the nonexistent land of Lexingtopia, there are 10 days in the year, and the Lexingtopian Math Society has 5 members. The probability that no two of the Lexingtopian Math Society's members share the same birthday can be written as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a+b$.

18. **[6]** Let $D(n)$ be the number of diagonals in a regular n-gon. Find

$$\sum_{n=3}^{26} D(n).$$

19. **[6]** Given a square $A_0B_0C_0D_0$ as shown below with side length 1, for all nonnegative integers n, construct points A_{n+1}, B_{n+1}, C_{n+1}, and D_{n+1} on A_nB_n, B_nC_n, C_nD_n, and D_nA_n, respectively, such that

$$\frac{A_nA_{n+1}}{A_{n+1}B_n} = \frac{B_nB_{n+1}}{B_{n+1}C_n} = \frac{C_nC_{n+1}}{C_{n+1}D_n} = \frac{D_nD_{n+1}}{D_{n+1}A_n} = \frac{3}{4}.$$



The sum of the series

$$\sum_{i=0}^{\infty} [A_iB_iC_iD_i] = [A_0B_0C_0D_0] + [A_1B_1C_1D_1] + [A_2B_2C_2D_2] \dots$$

where [**P**] denotes the area of polygon **P** can be written as $\frac{a}{b}$ where a and b are relatively prime positive integers. Find $a+b$.

20. **[6]** Let m and n be two real numbers such that

$$\frac{2}{n}+m=9$$
$$\frac{2}{m}+n=1$$

Find the sum of all possible values of m plus the sum of all possible values of n.

21. **[6]** Let $\sigma(x)$ denote the sum of the positive divisors of x. Find the smallest prime p such that

$$\sigma(p!) \geq 20 \cdot \sigma([p-1]!).$$

22. **[6]** Let $\triangle ABC$ be an isosceles triangle with $AB = AC$. Let M be the midpoint of side $\overline{AB}$. Suppose there exists a point X on the circle passing through points A, M, and C such that $BMCX$ is a parallelogram and M and X are on opposite sides of line BC. Let segments $\overline{AX}$ and $\overline{BC}$ intersect at a point Y. Given that $BY = 8$, find AY^2.

23. **[6]** Kevin chooses 2 integers between 1 and 100, inclusive. Every minute, Corey can choose a set of numbers and Kevin will tell him how many of the 2 chosen integers are in the set. How many minutes does Corey need until he is certain of Kevin's 2 chosen numbers?

24. **[6]** Evaluate

$$1^{-1}\cdot 2^{-1}+2^{-1}\cdot 3^{-1}+3^{-1}\cdot 4^{-1}+\cdots+(2015)^{-1}\cdot(2016)^{-1} \pmod{2017}.$$

25. **[6]** In scalene $\triangle ABC$, construct point D on the opposite side of AC as B such that $\angle ABD = \angle DBC = \angle BCA$ and $AD = DC$. Let I be the incenter of $\triangle ABC$. Given that $BC = 64$ and $AD = 225$, find BI.