13th Annual Spring Lexington Math Tournament - Guts Round - Part 1

Team Name: ______________________________

______________ 1. **[5]** A box contains 1 ball labelled W, 1 ball labelled E, 1 ball labelled L, 1 ball labelled C, 1 ball labelled O, 8 balls labelled M, and 1 last ball labelled E. One ball is randomly drawn from the box. The probability that the ball is labelled E is $\frac{1}{a}$. Find a.

______________ 2. **[5]** Let

$$\begin{aligned} G+E+N &= 7 \\ G+E+O &= 15 \\ N+T &= 22. \end{aligned}$$

Find the value of $T+O$.

______________ 3. **[5]** The area of $\triangle LMT$ is 22. Given that $MT = 4$ and that there is a right angle at M, find the length of LM.

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Team Name: ______________________________

______________ 4. **[5]** Kevin chooses a positive 2-digit integer, then adds 6 times its unit digit and subtracts 3 times its tens digit from itself. Find the greatest common factor of all possible resulting numbers.

______________ 5. **[5]** Find the maximum possible number of times circle D can intersect pentagon $GRASS'$ over all possible choices of points G, R, A, S, and S'.

______________ 6. **[5]** Find the sum of the digits of the integer solution to

$$(\log_2 x) \cdot (\log_4 \sqrt{x}) = 36.$$

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Team Name: ______________________________

______________ 7. **[6]** Given that x and y are positive real numbers such that $x^2 + y = 20$, the maximum possible value of $x + y$ can be written as $\frac{a}{b}$ where a and b are relatively prime positive integers. Find $a + b$.

______________ 8. **[6]** In $\triangle DRK$, $DR = 13$, $DK = 14$, and $RK = 15$. Let E be the point such that $ED = ER = EK$. Find the value of $\lfloor DE + RE + KE \rfloor$

______________ 9. **[6]** Subaru the frog lives on lily pad 1. There is a line of lily pads, numbered 2, 3, 4, 5, 6, and 7. Every minute, Subaru jumps from his current lily pad to a lily pad whose number is either 1 or 2 greater, chosen at random from valid possibilities. There are alligators on lily pads 2 and 5. If Subaru lands on an alligator, he dies and time rewinds back to when he was on lily pad number 1. Find how many times Subaru is expected to die before he reaches pad 7.

13th Annual Spring Lexington Math Tournament - Guts Round - Part 4

Team Name: ______________________________

______________ 10. **[6]** Find the sum of the following series:

$$\sum_{i=1}^{\infty} = \frac{\sum_{j=1}^{i} j}{2^i} = \frac{1}{2^1} + \frac{1+2}{2^2} + \frac{1+2+3}{2^3} + \frac{1+2+3+4}{2^4} + \ldots.$$

______________ 11. **[6]** Let $\varphi(x)$ be the number of positive integers less than or equal to x that are relatively prime to x. Find the sum of all x such that $\varphi(\varphi(x)) = x - 3$. Note that 1 is relatively prime to every positive integer.

______________ 12. **[6]** On a piece of paper, Kevin draws a circle. Then, he draws two perpendicular lines. Finally, he draws two perpendicular rays originating from the same point (an L shape). What is the maximum number of sections into which the lines and rays can split the circle?

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Team Name: ______________________________

______________ 13. **[7]** In quadrilateral $ABCD$, $\angle A = 90°$, $\angle C = 60°$, $\angle ABD = 25°$, and $\angle BDC = 5°$. Given that $AB = 4\sqrt{3}$, the area of quadrilateral $ABCD$ can be written as $a\sqrt{b}$. Find $10a + b$.

______________ 14. **[7]** The value of

$$\sum_{n=2}^{6} \left(\frac{n^4+1}{n^4-1}\right) - 2\sum_{n=2}^{6} \left(\frac{n^3-n^2+n}{n^4-1}\right)$$

can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $100m + n$.

______________ 15. **[7]** Positive real numbers x and y satisfy the following 2 equations.

$$x^{1+x^{1+x^{1+\ldots}}} = 8$$

$$\sqrt[24]{y + \sqrt[24]{y + \sqrt[24]{y + \ldots}}} = x$$

Find the value of $\lfloor y \rfloor$.

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Team Name: ______________________________

Déjà Vu?

______________ 16. **[7]** Given that x and y are positive real numbers such that $x^3 + y = 20$, the maximum possible value of $x + y$ can be written as $\frac{a\sqrt{b}}{c} + d$ where a, b, c, and d are positive integers such that $\gcd(a, c) = 1$ and b is square-free. Find $a + b + c + d$.

______________ 17. **[7]** In $\triangle DRK$, $DR = 13$, $DK = 14$, and $RK = 15$. Let E be the intersection of the altitudes of $\triangle DRK$. Find the value of $\lfloor DE + RE + KE \rfloor$

______________ 18. **[7]** Subaru the frog lives on lily pad 1. There is a line of lily pads, numbered 2, 3, 4, 5, 6, and 7. Every minute, Subaru jumps from his current lily pad to a lily pad whose number is either 1 or 2 greater, chosen at random from valid possibilities. There are alligators on lily pads 2 and 5. If Subaru lands on an alligator, he dies and time rewinds back to when he was on lily pad number 1. Find the expected number of jumps it takes Subaru to reach pad 7.

13th Annual Spring Lexington Math Tournament - Guts Round - Part 7

Team Name: ______________________

This set has problems whose answers depend on one another.

______ 19. **[8]** Let B be the answer to Problem 20 and let C be the answer to Problem 21. Given that $f(x) = x^3 - Bx - C = (x-r)(x-s)(x-t)$ where r, s, and t are complex numbers, find the value of $r^2 + s^2 + t^2$.

______ 20. **[8]** Let A be the answer to Problem 19 and let C be the answer to Problem 21. Circles ω_1 and ω_2 meet at points X and Y. Let point $P \neq Y$ be the point on ω_1 such that PY is tangent to ω_2, and let point $Q \neq Y$ be the point on ω_2 such that QY is tangent to ω_1. Given that $PX = A$ and $QX = C$, find XY.

______ 21. **[8]** Let A be the answer to Problem 19 and let B be the answer to Problem 20. Given that the positive difference between the number of positive integer factors of A^B and the number of positive integer factors of B^A is D, and given that the answer to this problem is an odd prime, find $\frac{D}{B} - 40$.

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Team Name: ______________________

______ 22. **[8]** Let $v_p(n)$ for a prime p and positive integer n output the greatest nonnegative integer x such that p^x divides n. Find

$$\sum_{i=1}^{50} \sum_{p=1}^{i} \binom{v_p(i)+1}{2},$$

where the inner summation only sums over primes p between 1 and i.

______ 23. **[8]** Let a, b, and c be positive real solutions to the following equations.

$$\frac{2b^2+2c^2-a^2}{4} = 25$$
$$\frac{2c^2+2a^2-b^2}{4} = 49$$
$$\frac{2a^2+2b^2-c^2}{4} = 64$$

The area of a triangle with side lengths a, b, and c can be written as $\frac{x\sqrt{y}}{z}$ where x and z are relatively prime positive integers and y is square-free. Find $x + y + z$.

______ 24. **[8]** Alan, Jiji, Ina, Ryan, and Gavin want to meet up. However, none of them know when to go, so they each pick a random 1 hour period from 5 AM to 11 AM to meet up at Alan's house. Find the probability that there exists a time when all of them are at the house at one time.

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______________ 25. **[8]** Let n be the number of registered participants in this LMT. Estimate the number of digits of $\left[\binom{n}{2}\right]!$ in base 10. If your answer is A and the correct answer is C, then your score will be

$$\left\lfloor \max\left(0, 20 - \left|\ln\left(\frac{A}{C}\right)\cdot 5\right|\right)\right\rfloor.$$

______________ 26. **[8]** Let γ be the minimum value of x^x over all real numbers x. Estimate $\lfloor 10000\gamma \rfloor$. If your answer is A and the correct answer is C, then your score will be

$$\left\lfloor \max\left(0, 20 - \left|\ln\left(\frac{A}{C}\right)\cdot 5\right|\right)\right\rfloor.$$

______________ 27. **[8]** Let

$$E = \log_{13} 1 + \log_{13} 2 + \log_{13} 3 + \cdots + \log_{13} 513513.$$

Estimate $\lfloor E \rfloor$. If your answer is A and the correct answer is C, your score will be

$$\left\lfloor \max\left(0, 20 - \left|\ln\left(\frac{A}{C}\right)\cdot 5\right|\right)\right\rfloor.$$

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