Guts Round Solutions

Lexington High School

May 9th, 2021;.

12th Annual Lexington Math Tournament - Guts Round - Part 1

Team Name: ______________________________

______________ 1. **[5]** How many ways are there to arrange the letters in the word NEVERLAND such that the 2 N's are adjacent and the two E's are adjacent? Assume that letters that appear the same are not distinct.

Proposed by Kira Tang

Solution. $\boxed{5040}$

We can group the two N's together and the two E's together. This gives us 2 pairs and 5 individual letters. Remembering that the order of the N's and E's does not matter, there are $7! = 5040$ ways to arrange the letters. □

______________ 2. **[5]** In rectangle $ABCD$, E and F are on AB and CD, respectively such that $DE = EF = FB$ and $\angle CDE = 45°$. Find $AB + AD$ given that AB and AD are relatively prime positive integers.

Proposed by Ada Tsui

Solution. $\boxed{4}$

We begin by angle-chasing. We know that $\angle CDE = 45°$ from the problem and $\angle A = \angle B = \angle C = \angle D = 90°$ due to $ABCD$ being a rectangle. Then $\angle = ADE = 45°$, $\angle AED = 45°$ due to the sum of angles in a triangle, and $\angle DFE = 45°$. □

______________ 3. **[5]** Maisy Airlines sees n takeoffs per day. Find the minimum value of n such that there must exist two planes that take off within a minute of each other.

Proposed by Ephram Chun

Solution. $\boxed{1441}$

There are $24 * 60 = 1440$ minutes in a day. Thus by Pigeon Hole Principle the minimum value of n is $\boxed{1441}$. □

12th Annual Lexington Math Tournament - Guts Round - Part 2

Team Name: ______________________________

______________ 4. **[5]** Nick is mixing two solutions. He has 100 mL of a solution that is 30% X and 400 mL of a solution that is 10% X. If he combines the two, what percent X is the final solution?

Proposed by Sammy Charney

Solution. $\boxed{14}$

We calculate how many mL of X are in each solution. There are 30 mL in the first solution, and 40 mL in the second solution. Therefore, there are 70 mL of X in the final solution which is 500 mL, meaning it is $\boxed{14}$% X. $\square$

______________ 5. **[5]** Find the number of ordered pairs (a, b), where a and b are positive integers, such that

$$\frac{1}{a}+\frac{2}{b}=\frac{1}{12}.$$

Proposed by Brian Yu

Solution. $\boxed{18}$

Multiplying the equation by $12ab$ gets that we have $12b+24a = ab$ so $(a-12)(b-24) = 288$. As a result, we see that $a-12$ just has to be a positive integer factor of 288 and $b-24$ automatically is too, hence our answer is just the number of positive divisors of 288. Note that $288 = 2^5 \cdot 3^2$ so the number of factors if $(5+1)(2+1) = \boxed{18}$. $\square$

______________ 6. **[5]** 25 balls are arranged in a 5 by 5 square. Four of the balls are randomly removed from the square. Given that the probability that the square can be rotated 180° and still maintain the same configuration can be expressed as $\frac{m}{n}$, where m and n are relatively prime, find $m+n$.

Proposed by Ada Tsui

Solution. $\boxed{578}$

$\frac{\binom{12}{2}}{\binom{25}{4}} = \frac{3}{575}$. $3+575 = 578$. $\square$

12th Annual Lexington Math Tournament - Guts Round - Part 3

Team Name: ________________________________

________________ 7. **[6]** Maisy the ant is on corner A of a $13 \times 13 \times 13$ box. She needs to get to the opposite corner called B. Maisy can only walk along the surface of the cube and takes the path that covers the least distance. Let C and D be the possible points where she turns on her path. Find

$$AC^2 + AD^2 + BC^2 + BD^2 - AB^2 - CD^2.$$

Proposed by Ephram Chun

Solution. $\boxed{}$

Due to the problem being very ambiguous, all teams who inputted answers received points. □

________________ 8. **[6]** Maisyton has recently built 5 intersections. Some intersections will get a park and some of those that get a park will also get a chess school. Find how many different ways this can happen.

Proposed by Ephram Chun

Solution. $\boxed{243}$

Each intersection has 3 different outcomes. Thus our answer is $3^5 = \boxed{243}$. □

________________ 9. **[6]** Let $f(x) = 2x - 1$. Find the value of x that minimizes

$$|f(f(f(f(f(x))))) - 2020|.$$

Proposed by Ephram Chun

Solution. $\boxed{2051/32}$

$$f^5(x) = 32x - 31$$

Our answer is $\boxed{\frac{2051}{32}}$. We also accepted 64 as an answer, because it was said that all answers were integers beforehand. □

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12th Annual Lexington Math Tournament - Guts Round - Part 4

Team Name: ______________________________

______________ 10. **[6]** Triangle ABC is isosceles, with $AB = BC > AC$. Let the angle bisector of $\angle A$ intersect side $\overline{BC}$ at point D, and let the altitude from A intersect side $\overline{BC}$ at point E. If $\angle A = \angle C = x°$, then the measure of $\angle DAE$ can be expressed as $(ax-b)°$, for some constants a and b. Find ab.

Proposed by Aditya Rao

Solution. $\boxed{135}$

Note that $\angle DAC = \left(\frac{x}{2}\right)^\circ$ because $\overline{AD}$ bisects $\angle A$. Next, we have that

$$\angle ADC = 180^\circ - \angle DAC - \angle C = 180^\circ - \left(\frac{3x}{2}\right)^\circ$$

by using the fact that the sum of the interior angles in $\triangle ADC$ is $180°$. Finally, we have that

$$\angle DAE = 90^\circ - \angle ADE = 90^\circ - \left(180^\circ - \left(\frac{3x}{2}\right)^\circ\right) = \left(\frac{3x}{2} - 90\right)^\circ.$$

With $a = \frac{3}{2}$ and $b = 90$, we have that $ab = \boxed{135}$. □

______________ 11. **[6]** Maisy randomly chooses 4 integers w, x, y, and z, where $w, x, y, z \in \{1, 2, 3, \dots, 2019, 2020\}$. Given that the probability that $w^2 + x^2 + y^2 + z^2$ is not divisible by 4 is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

Proposed by Ephram Chun

Solution. $\boxed{15}$

We proceed with complementary counting. We see that if a number is odd then the square of that number will always be $1 \pmod 4$, and when a number is even then the square of that number will always be $0 \pmod 4$. That means that w, x, y, z must either be all even or all odd. The probability that w, x, y, z are all odd or even is $2\left(\frac{1}{2}\right)^4 = \frac{1}{8}$. Thus, our answer is $1 - \frac{1}{8} = \frac{7}{8} \implies \boxed{15}$. □

______________ 12. **[6]** Evaluate

$$-\log_4\left(\log_2\left(\sqrt{\sqrt{\sqrt{\cdots\sqrt{16}}}}\right)\right),$$

where there are 100 square root signs.

Proposed by Powell Zhang

Solution. $\boxed{49}$

This is just $-\log_4(\log_2(2^{\frac{4}{2^{100}}})) = -\log_4(\log_2(2^{\frac{1}{2^{98}}})) = -\log_4\left(\frac{1}{2^{98}}\right) = \boxed{49}$ □

12th Annual Lexington Math Tournament - Guts Round - Part 5

Team Name: ______________________________

______________ 13. **[7]** Pieck the Frog hops on Pascal's Triangle, where she starts at the number 1 at the top. In a hop, Pieck can hop to one of the two numbers directly below the number she is currently on with equal probability. Given that the expected value of the number she is on after 7 hops is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

Proposed by Steven Yu

Solution. $\boxed{445}$

We note that on the row that the frog will end up on, we will have $\binom{7}{0}, \binom{7}{1}, \dots, \binom{7}{7}$ be the possible values. To get to $\binom{7}{0}$, we need to go left 7 times and right 0 times, so we have $\binom{7}{7}$ ways to get to this value. For $\binom{7}{1}$, we need to go left 6 times and right 1 time, so we have $\binom{7}{6}$ ways to get to this value. This continues on, so that for $\binom{7}{k}$ we have $\binom{7}{7-k}$ ways to get to this value. Thus, since we have 2^7 paths we could traverse, our final expected value is

$$\frac{\binom{7}{0}\cdot\binom{7}{7}+\binom{7}{1}\cdot\binom{7}{6}+\cdots+\binom{7}{7}\cdot\binom{7}{0}}{2^7}=\frac{\binom{14}{7}}{2^7}=\frac{429}{16},$$

where we used Vandermonde's identity to compute our sum of products of binomials. Thus, our answer is $m+n=429+16=\boxed{445}$. □

______________ 14. **[7]** Maisy chooses a random set (x, y) that satisfies

$$x^2+y^2-26x-10y\le 482.$$

The probability that $y>0$ can be expressed as $\frac{A\pi-B\sqrt{C}}{D\pi}$. Find $A+B+C+D$.

Proposed by Ephram Chun

Solution. $\boxed{}$

Due to the problem having a typo, all teams who inputted answers received points. □

______________ 15. **[7]** 6 points are located on a circle. How many ways are there to draw any number of line segments between the points such that none of the line segments overlap and none of the points are on more than one line segment? (It is possible to draw no line segments).

Proposed by Brian Yu

Solution. $\boxed{51}$

We can simply do careful casework. For 0 lines, we have 1 configuration. For 1 line, we have $\binom{6}{2}=15$ configurations. For 2 lines, we have 30 configurations. For 3 lines, we have 5 configurations, so in total we have $\boxed{51}$ configurations. □

12th Annual Lexington Math Tournament - Guts Round - Part 6

Team Name: ________________________________

________________ 16. **[7]** Find the number of 3 by 3 grids such that each square in the grid is colored white or black and no two black squares share an edge.

Proposed by Brian Yu

Solution. $\boxed{63}$

We do casework on the number of black squares present.

0 black squares: 1 possible grid coloring.

1 black square: 9 possible grid colorings.

2 black squares: $\binom{9}{2}$ arrangements, minus 12 ways for the arrangement to have a shared edge. So, 24 possible grid colorings.

3 black squares: We have several cases. We can have (1) three squares bordering the center, for 4 colorings, (2) choosing any three squares out of the four corners and the center for $\binom{5}{3} = 10$ colorings, (3) choosing two corners on the same side and the square adjacent to the center on the opposite side for 4 colorings, and (4) one corner and the opposite two squares bordering the center, for 4 more colorings. In total we have 22 colorings here.

4 black squares: We have several cases. We can (1) have all four squares bordering the center for 1 coloring, (2) choose any four squares out of the four corners and the center for $\binom{5}{4} = 5$ colorings. In total we have 6 colorings here.

5 black squares: 1 possible coloring consisting of the four corners and the center.

In total, we have $1+9+24+22+6+1 = \boxed{63}$ possible colorings. □

________________ 17. **[7]** Let ABC be a triangle with side lengths $AB = 20$, $BC = 25$, and $AC = 15$. Let D be the point on BC such that $CD = 4$. Let E be the foot of the altitude from A to BC. Let F be the intersection of AE with the circle of radius 7 centered at A such that F is outside of triangle ABC. DF can be expressed as $\sqrt{m}$, where m is a positive integer. Find m.

Proposed by David Sun

Solution. $\boxed{386}$

We can form a 12-16-20 right triangle ABE, while we can form a 9-12-15 triangle ACE. Therefore, we can conclude that DE = 5. Drawing AE beyond triangle ABC such that it intersects the circle of radius 7 centered at A, we can conclude that EF = 19. Therefore, we can find that DF $= \sqrt{5^2+19^2} = \sqrt{386}, and$m = 386. □

________________ 18. **[7]** Bill and Frank were arrested under suspicion for committing a crime and face the classic Prisoner's Dilemma. They are both given the choice whether to rat out the other and walk away, leaving their partner to face a 9 year prison sentence. Given that neither of them talk, they both face a 3 year sentence. If both of them talk, they both will serve a 6 year sentence. Both Bill and Frank talk or do not talk with the same probabilities. Given the probability that at least one of them talks is $\frac{11}{36}$, find the expected duration of Bill's sentence in months.

Proposed by Sammy Charney

Solution. $\boxed{42}$

The probability neither of them talk is $\frac{25}{36}$, so each person doesn't talk with probability $\frac{5}{6}$. Therefore, accounting for all scenarios, the expected duration of Bill's sentence is $\frac{5}{36}(0)+\frac{25}{36}(3)+\frac{1}{36}(6)+\frac{5}{36}(9) = \frac{7}{2}$ years, which is $\boxed{42}$ months. □

12th Annual Lexington Math Tournament - Guts Round - Part 7

Team Name: ______________________________

______________ 19. **[8]** Rectangle $ABCD$ has point E on side $\overline{CD}$. Point F is the intersection of $\overline{AC}$ and $\overline{BE}$. Given that the area of $\triangle AFB$ is 175 and the area of $\triangle CFE$ is 28, find the area of $ADEF$.

Proposed by Powell Zhang

Solution. $\boxed{217}$

$\triangle AFB$ and $\triangle CFE$ are similar and have areas in a ratio of $25:4$, so their sides have the ratio $5:2$. Thus, the area of $\triangle ABC$ will be area $\triangle AFB * \frac{(2+5)}{5}$ or $135 * \frac{7}{5} = 245$. The area of $ADEF$ is then just area $\triangle ADC$ (which equals area $\triangle ABC$) - area $\triangle CFE = 245 - 28 = 217$. □

______________ 20. **[8]** Real numbers x, y, and z satisfy the system of equations

$$5x + 13y - z = 100,$$
$$25x^2 + 169y^2 - z^2 + 130xy = 16000,$$
$$80x + 208y - 2z = 2020.$$

Find the value of xyz.

Proposed by Ephram Chun

Solution. $\boxed{}$

Due to the problem having infinitely many solutions, all teams who inputted answers received points. □

______________ 21. **[8]** Bob is standing at the number 1 on the number line. If Bob is standing at the number n, he can move to $n+1$, $n+2$, or $n+4$. In how many different ways can he move to the number 10?

Proposed by Brian Yu

Solution. $\boxed{96}$

Let S_n denote the number of ways that Bob can reach the number n on the number line; thus, we want to find S_{10}. Note that we can first fine $S_2 = 1$, since we need to go one step, $S_3 = 2$, since we can either go $1 \to 2 \to 3$ or $1 \to 3$, and then $S_4 = 3$, since we can go $1 \to 2 \to 3 \to 4$, $1 \to 2 \to 4$, or $1 \to 3 \to 4$. Then, from here, we have that $S_k = S_{k-4} + S_{k-2} + S_{k-1}$, since those are all the places we could be coming from. By using recursion in this fashion, we are able to work out that $S_{10} = \boxed{96}$. □

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12th Annual Lexington Math Tournament - Guts Round - Part 8

Team Name: ______________________________

______________ 22. **[8]** A sequence $a_1, a_2, a_3, \ldots$ of positive integers is defined such that $a_1 = 4$, and for each integer $k \geq 2$,

$$2(a_{k-1} + a_k + a_{k+1}) = a_k a_{k-1} + 8.$$

Given that $a_6 = 488$, find $a_2 + a_3 + a_4 + a_5$.

Proposed by Taiki Aiba

Solution. $\boxed{92}$

Rearrange the given equation to

$$2(a_{k+1} - 2) = (a_k - 2)(a_{k-1} - 2).$$

Let $b_n = a_n - 2$ for all positive integers n. We have the equation

$$2b_{k+1} = b_k b_{k-1}.$$

Let $b^2 = d$. Listing out the first few terms of b_n, we get

$$2, d, d, \frac{d^2}{2}, \frac{d^3}{4}, \frac{d^5}{16}.$$

Solving $\frac{d^5}{16} = 486$, we get $d = 6$. Then, our list becomes

$$2, 6, 6, 18, 54, 486,$$

so our a_n list is

$$4, 8, 8, 20, 56, 488.$$

Our requested sum is $8 + 8 + 20 + 56 = \boxed{92}$. □

______________ 23. **[8]** $\overline{PQ}$ is a diameter of circle ω with radius 1 and center O. Let A be a point such that AP is tangent to w. Let γ be a circle with diameter AP. Let A' be where AQ hits the circle with diameter AP and A'' be where AO hits the circle with diameter OP. Let $A'A''$ hit PQ at R. Given that the value of the length RA' is always less than k and k is minimized, find the greatest integer less than or equal to $1000k$.

Proposed by Kevin Zhao

Solution. $\boxed{1333}$

First, we realize that playing around with right angles,

$$\angle PA'A = \angle PA''A = 90^\circ$$

and so $AA'A''P$ is a cyclic quadrilateral. Now, we angle chase and see

$$\angle AA''A' = \angle APA' = \angle PQA' \to \angle OQA' + \angle A'A''O = 180^\circ$$

meaning that $A'A''OQ$ is cyclic. We then note that

$$(1 - RO)^2 = RP^2 = RQ' \cdot RA'' = RO \cdot OQ = RO(RO + 1)$$

which means that

$$RO^2 - 2RO + 1 = RO^2 + RO \to RO = \frac{1}{3}$$

so R is fixed. Since A' is on the circle with diameter QP, we see that the furthest point from R on the such circumcircle would be the point on RO such that it is on the circumcircle and on the other side of O as R. This point is also Q. Checking, we can have that A is arbitrarily far from P, getting that A' is arbitrarily close to Q, and never exactly Q. Hence, our value of k will be $RQ = \frac{4}{3}$ and $\lfloor 1000k \rfloor = \boxed{1333}$. □

24. **[8]** You have cards numbered $1, 2, 3, \ldots, 100$ all in a line, in that order. You may swap any two adjacent cards at any time. Given that you make $\binom{100}{2}$ total swaps, where you swap each distinct pair of cards exactly once, and do not do any swaps simultaneously, find the total number of distinct possible final orderings of the cards.

Proposed by Richard Chen

Solution. $\boxed{1}$

We use induction. With 2 cards, the only possible outcome at the end is $2, 1$, and with 3 cards, we can check swapping both $(1,2) \to (1,3) \to (2,3)$ and $(2,3) \to (1,3) \to (1,2)$ to see that the only possible outcome at the end is $3, 2, 1$. Assume that for k cards, the only possible end configuration is $k, k-1, k-2, \ldots, 3, 2, 1$. Observe that for any two cards $a < b$, at the instance that we swap cards a and b, b necessarily needs to move to the left, and a needs to move to the right. Otherwise, if b moves to the right and a to the left, that means that somehow a had to end up on the right of b at some point, which can only happen if they swapped with each other before. So, with $k+1$ cards, the card $k+1$ will necessarily move k spots to the left as it swaps with $1, 2, \ldots, k$, so that the final order is $k+1, k, k-1, \ldots, 1$. So, it will always have to be completely reversed, and there is only $\boxed{1}$ possible end configuration. $\square$

12th Annual Lexington Math Tournament - Guts Round - Part 9

Team Name: ______________________________

______________ 25. **[9]** Let a, b, and c be positive numbers with $a+b+c=4$. If $a,b,c,\leq 2$ and

$$M=\frac{a^3+5a}{4a^2+2}+\frac{b^3+5b}{4b^2+2}+\frac{c^3+5c}{4c^2+2},$$

then find the maximum possible value of $\lfloor 100M \rfloor$.

Proposed by Kevin Zhao

Solution. $\boxed{300}$

Note that $a^3-4a^2+5a-2=(x-1)^2(x-2)<0$ if $0\leq a\leq 2$ meaning that $a^3+5a<4a^2+2$ and so dividing, since both sides are positive, $\frac{a^3+5a}{4a^2+2}<1$. Applying this to b and c too, we see that $M\leq 1+1+1=3$. Note that $(a,b,c)=(1,1,2)$ yields $M=3$, so $\lfloor 100M\rfloor \leq \boxed{300}$. $\square$

______________ 26. **[9]** In $\triangle ABC$, $AB=15$, $AC=16$, and $BC=17$. Points E and F are chosen on sides AC and AB, respectively, such that $CE=1$ and $BF=3$. A point D is chosen on side BC, and let the circumcircles of $\triangle BFD$ and $\triangle CED$ intersect at point $P\neq D$. Given that $\angle PEF=30^\circ$, the length of segment PF can be expressed as $\frac{m}{n}$. Find $m+n$.

Proposed by Andrew Zhao

Solution. $\boxed{79}$

By Law of Cosines on $\triangle ABC$, we have $17^2=15^2+16^2-2(15)(16)(\cos\angle BAC)$, so $\cos\angle BAC=\frac{2}{5}$. Law of Cosines on $\triangle AEF$ gives $EF^2=12^2+15^2-2(12)(15)\left(\frac{2}{5}\right)$, so $EF=15$. By Miquel's Point, $AEPF$ is cyclic, so $\cos\angle EPF=\cos\angle EAF=\frac{2}{5}$, which means $\sin\angle EPF=\sqrt{1-(2/5)^2}=\frac{\sqrt{21}}{5}$. By Law of Sines on $\triangle EPF$, we have $\frac{PF}{\sin\angle PEF}=\frac{EF}{\sin\angle EPF}\implies PF=\frac{(EF)(\sin\angle PEF)}{\sin\angle EPF}=\frac{(15)(1/2)}{2/5}=\frac{75}{4}$, so the answer is 79 $\square$

______________ 27. **[9]** Arnold and Barnold are playing a game with a pile of sticks with Arnold starting first. Each turn, a player can either remove 7 sticks or 13 sticks. If there are fewer than 7 sticks at the start of a player's turn, then they lose. Both players play optimally. Find the largest number of sticks under 200 where Barnold has a winning strategy.

Proposed by Powell Zhang

Solution. $\boxed{186}$

Let n be the starting number of sticks. Barnold clearly wins if $n<7$. For every $n>7$, we can create a recursive formula where Arnold wins if $n-7$ or $n-13$ is a win for Barnold, else Barnold wins. Writing this out, we see that the winners repeat with Arnold winning 13, then Barnold winning 7, then Arnold winning 13. Barnold wins if $20z<=n<=20z+6$ where z is an integer, thus the largest $n<200$ where Barnold wins is $\boxed{186}$. $\square$

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12th Annual Lexington Math Tournament - Guts Round - Part 10

Team Name: ________________________________

________________ 28. **[11]** Let a, b, and c be positive real numbers such that

$$\log_2(a) - 2 = \log_3(b) = \log_5(c) \quad \text{and} \quad a + b = c.$$

What is $a+b+c$?

Proposed by Powell Zhang

Solution. $\boxed{50}$

This is essentially asking to find x such that $2^{x+2} + 3^x = 5^x$. It is relatively easy to see that $x = 2$ works. From there, we can see that $-\infty < \mathrm{x} < 2$ is greater than 0 and $2 < \mathrm{x} < \infty$ is less than 0 because 5^x grows faster than the left side so $x = 2$ is the only solution that works. Thus $a = 16, b = 9$, and $c = 25$ so $a + b + c = \boxed{50}$. □

________________ 29. **[11]** Two points, $P(x, y)$ and $Q(-x, y)$ are selected on parabola $y = x^2$ such that $x > 0$ and the triangle formed by points P, Q, and the origin has equal area and perimeter. Find y.

Proposed by Hannah Shen

Solution. $\boxed{8}$

The area of this triangle can be expressed as $x * x^2$ or x^3, and its perimeter can be expressed as $2x + 2\sqrt{x^2 + x^4}$ or $2x + 2x\sqrt{1 + x^2}$. To find y, we solve the following equation:

$$x^3 = 2x + 2x\sqrt{1+x^2}$$
$$x^2 = 2 + 2\sqrt{1+x^2}$$
$$x^2 - 2 = 2\sqrt{1+x^2}$$
$$x^4 - 4x^2 + 4 = 4 + 4x^2$$
$$x^4 - 8x^2 = 0$$
$$(x^2)(x^2 - 8) = 0$$

Because we know $y = x^2$, our two solutions are $y = 0$ and $y = 8$, of which only the latter is applicable. Therefore, $y = \boxed{8}$. □

________________ 30. **[11]** 5 families are attending a wedding. 2 families consist of 4 people, 2 families consist of 3 people, and 1 family consists of 2 people. A very long row of 25 chairs is set up for the families to sit in. Given that all members of the same family sit next to each other, let the number of ways all the people can sit in the chairs such that no two members of different families sit next to each other be n. Find the number of factors of n.

Proposed by Sammy Charney

Solution. $\boxed{480}$

We plan on placing the first person in each family and then inserting the rest of the family after them and dividers between families. We have 5 families, totaling 16 people, so there are 11 family members to be inserted along with 4 dividers. Thus, we "send 15 chairs to the bathroom, " removing them from where the first person in the family can be placed. This means we have $\binom{10}{5}$ ways to place the first person of each family. We must then order the families and order the members of each family, giving us the total number of ways is $n = \binom{10}{5} \cdot 5! \cdot 4!^2 \cdot 3!^2 \cdot 2!^1 = 2^{14} \cdot 3^7 \cdot 5 \cdot 7$. This has $15 \cdot 8 \cdot 2 \cdot 2 = \boxed{480}$ factors. □

12th Annual Lexington Math Tournament - Guts Round - Part 11

Team Name: ______________________________

______________ 31. **[13]** Let polynomial $P(x) = x^3 + ax^2 + bx + c$ have (not neccessarily real) roots r_1, r_2, and r_3. If $2ab = a^3 - 20 = 6c - 21$, then the value of $|r_1^3 + r_2^3 + r_3^3|$ can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find the value of $m + n$.

Proposed by Kevin Zhao

Solution. $\boxed{21}$

Note that

$$r_1^3 + r_2^3 + r_3^3 - 3r_1r_2r_3 = ((r_1 + r_2 + r_3)^2 - 3(r_1r_2 - r_2r_3 - r_3r_1))(r_1 + r_2 + r_3)$$

and now, getting that due to Vietas, we have $r_1r_2r_3 = -c$, $r_1 + r_2 + r_3 = -a$, and $r_1r_2 + r_2r_3 + r_3r_1 = b$. Plugging in gets that we have

$$r_1^3 + r_2^3 + r_3^3 + 3c = (a^2 - 3b)(-a) = 3ab - a^3.$$

Rearranging $2ab = a^3 - 20$, we see that $ab - 20 = 3ab - a^3$, and since $2ab = 6c - 21$, then $ab = 3c - \frac{21}{2}$ and $3c = \frac{21}{2} + ab$. Now, plugging in these such equations, we get that

$$r_1^3 + r_2^3 + r_3^3 + \left(ab - \frac{21}{2}\right) = ab - 20 \to |r_1^3 + r_2^3 + r_3^3| = \frac{19}{2}.$$

Our answer is thus $19 + 2 = \boxed{21}$. □

______________ 32. **[13]** In acute $\triangle ABC$, let H, I, O, and G be the orthocenter, incenter, circumcenter, and centroid of $\triangle ABC$, respectively. Suppose that there exists a circle ω passing through B, I, H, and C, the circumradius of $\triangle ABC$ is 312, and $OG = 80$. Let H', distinct from H, be the point on ω such that $\overline{HH'}$ is a diameter of ω. Given that lines $H'O$ and BC meet at a point P, find the length OP.

Proposed by Kevin Zhao

Solution. $\boxed{169}$

Note that first, we see that letting M_A be the midpoint of BC, then since $OM_A || AH$, then because A, G, and M_A are collinear, we have that $\frac{GO}{GH} = \frac{GM_A}{GA} = \frac{1}{2}$ (it is well known that $GA = 2 \cdot GM_A$). Now, we see that as a result, $HO = GO + GH = 3 \cdot GO = 240$.

We also note that since B, I, H, and C are concyclic, then $\angle BHC = \angle BIC$. We calculate the angles and see that $\angle BHC = 180^\circ - \angle A$ and

$$\angle BIC = 180^\circ - \frac{\angle B}{2} - \frac{\angle C}{2} = 90^\circ + \frac{\angle A}{2} \to 180^\circ - \angle A = 90^\circ + \frac{\angle A}{2} \to \angle A = 60^\circ.$$

This now shows us that $\angle BHC = 120^\circ$, and because $\angle BOC = 2\angle BAC = 120^\circ$, then B, O, H, and C are concyclic, too.

Now, note that because $\angle BAC = 60^\circ$, then since $\angle CH'B = 180^\circ - \angle BHC = 60^\circ$, then we have that the letting the circumcenter of BHC be O_1, then $\angle CO_1B = 2\angle CH'B = 120^\circ$ meaning that O_1 is on the circumcircle of $\triangle ABC$. So,

$$OA = OB = OC = OO_1 = O_1H = O_1H' = 312 \to HH' = 624$$

and now

$$H'O = \sqrt{H'H^2 - HO^2} = \sqrt{624^2 - 240^2} = 576.$$

Because BC is the radical axis of ω and the circumcircle of $\triangle ABC$, then we can invert about O with radius OA. This takes lines to circles passing through the center of inversion and vice versa, meaning that since B goes to itself and C does too, then H' goes to P and vice versa since P is on BC and H' is on ω. Thus, $OP \cdot OH' = OAO_1^2$ because the product of any center of inversion to a point before and after inversion is the same as the radius of inversion. So, we have $OP \cdot 576 = 312^2 \to OP = \boxed{169}$. □

33. **[13]** Find the number of ordered quadruples (x, y, z, w) such that $0 \le x, y, z, w \le 1000$ are integers and

$$x! + y! = 2^z \cdot w!$$

holds (Note: $0! = 1$).

Proposed by Kevin Zhao

Solution. $\boxed{1039}$

First, we can WLOG that $x < y$, and manage $x = y$ later. So, taking v_2 gets that $v_2(x! + y!) = v_2(2^z \cdot w!)$.

First, we take the $z = 0$ case. We have $x! + y! = w!$ which means that since $(k+1)! > k! \cdot 2$ for all $k > 1$, then since $x! + y! \le w!$, we see that $w > x$ and $w > y$ because $(k+1)! \ge k! > 0$ for all nonnegative k. Hence, now we just have to take the case of $w = 2$ because if $w = 0$ or $w = 1$, then $w! = 1$ and since $x!, y! \ge 1$, we have a contradiction. If $w = 2$, then we have $x! + y! = 2 \to (0,1), (1,0)$ work, after reaching to all $x \neq y$.

Now, we take $z \neq 0$. $v_2(w!) < v_2(2^z \cdot w!) = v_2(x! + y!)$, then if $y \ge x + 2$, we note that $v_2(x! + y!) = v_2(x!)$ meaning that $v_2(w!) < v_2(x!)$. So, we see that in this case, $x > w$. Noticing this, since $y > x$, we let $x = w + a$ and $y = x + b$ where a and b are positive integers, which gets us that $(w+a+b)! + (w+a)! = 2^z \cdot w!$ and since $z \neq 0$, then 2^z is even and $2^z = \frac{(w+a+b)!+(w+a)!}{w!}$ is even. This means that either both $\frac{(w+a+b)!}{w!}$ and $\frac{(w+a)!}{w!}$ are odd or even. Since $w + a + b \ge w + 1 + 1 = w + 2$, then $\frac{(w+a+b)!}{w!}$ is even and $\frac{(w+a)!}{w!}$ is too. But, because now,

$$2^z = \frac{(w+a+b)! + (w+a)!}{w!} = ((w+a+1)\dots(w+a+b)+1)(w+1)\dots(w+a),$$

then we see that we now have two cases:

1) at least one of $w+1$ and $w+2$ is not a power of 2, meaning that $w+1$ must be a power of 2 and $w+2$ not. This means that $a = 1$ too, and $2^z = (w+1)((w+2)\dots(w+1+b)+1)$ so $w+1$ is a power of 2 and we now see if $b \ge 2$, then $(w+a+1)\dots(w+a+b)+1$ is odd and so $b = 1$ meaning that $2^z = (w+1)(w+3) \to w = 1$ and $z = 3$ is our only solution, with $(x, y) = (2,3)$ and $(3,2)$. However, the difference is not at least two. 2) both $w+1$ and $w+2$ are powers of 2. Note that $w+3$ can never be a power of two, as a result. Hence, $w = 0$ and $a = 2$. So, we get $2! + y! = 2^z \to v_2(y) = 2$ or $z \le 1$. If $z = 0$, then we get $y! = -1$ and no solutions; if $z = 1$ then $y! = 0$ also has no solutions; thus, $v_2(y) = 1$ and $y > x \to y = 3$. However, the difference is not at least two.

Now, observe that we still have the $y = x+1$ case. Thus, $x!(x+2) = 2^z \cdot w!$. Hence, we see if $w < x$ then $x(x+2)$ is a power of w and again, x and $x+2$ are powers of 2; since $w < x$ then $x = 2$, so $(2,3,3,1), (3,2,3,1), (2,3,3,0)$, and $(3,2,3,0)$ are thus solutions and no other is valid. $x = 0$ - a special case - gets that $y! + 1 = 2^z \cdot w!$ and taking mod 2, we check that $z = 0$ gets $y = 0, 1$ to work, and if $x > 0$ then $y = 0, 1$ since $y!$ is then odd. If $w = x$ then $x + 2 = 2^z$ and so we have $(2^z - 2, 2^z - 1, z, 2^z - 2)$ and $(2^z - 1, 2^z - 2, z, 2^z - 2)$ are solution sets, after reaching to all $x \neq y$. If $x = y$, we see that $2x! = 2^z \cdot w!$ and $(x, x, 1, x)$ is a solution set, after reaching to all $x \neq y$. In addition, if w is a power of 2, we can decrease w by 1 which gets another solution set $(2^k, 2^k, k+1, 2^k - 1)$ where $1 \le k \le 9$

So, taking all the solutions and extending to $x > y$, we see that our possible sets are $(x, x, 1, x)$, $(2^z - 2, 2^z - 1, z, 2^z - 2)$, and $(2^z - 1, 2^z - 2, z, 2^z - 2)$. Our special case extra solutions founded (and unioned with the normal ones) are $(2,3,3,1)$, $(3,2,3,1)$, $(2,3,3,0)$, $(3,2,3,0)$, $(0,1,0,2)$, $(1,0,0,2)$, $(1,1,0,2)$, $(1,1,1,0)$, $(0,1,1,1)$, $(1,0,1,1)$, and $(0,0,0,2)$ - that makes eleven. $(x, x, 1, x)$ gives 1001 solutions, $(2^z - 2, 2^z - 1, z, 2^z - 2)$ gives 9 solutions, $(2^z - 1, 2^z - 2, z, 2^z - 2)$ gives another 9 solutions, and $(2^k, 2^k, 2k, 2^k - 1)$ gives nine more. Our answer is $11 + 1001 + 9 + 9 + 9 = \boxed{1039}$ solutions. □

12th Annual Lexington Math Tournament - Guts Round - Part 12

Team Name: ________________________

________ 34. **[15]** Let Z be the product of all the answers from the teams for this question. Estimate the number of digits of Z. If your estimate is E and the answer is A, your score for this problem will be

$$\max\left(0, \left\lceil 15 - |A-E| \right\rceil\right)$$

Your answer must be a positive integer.

Proposed by Ephram Chun

Solution. $\boxed{5544}$

Basically we got 1 submission that was 5168 digits long consisting of a lot of 6's and 9's. Then by simple multiplication the number of digits in our total product was $\boxed{5544}$ □

________ 35. **[15]** Let N be number of ordered pairs of positive integers (x, y) such that $3x^2 - y^2 = 2$ and $x < 2^{75}$. Estimate N. If your estimate is E and the answer is A, your score for this problem will be

$$\max\left(0, \left\lceil 15 - 2|A-E| \right\rceil\right)$$

Proposed by Kevin Zhao

Solution. $\boxed{40}$

$(1,1)$ is the smallest solution, so we can then apply Pell's Equations. Note that our solutions are $O((2+\sqrt{3})^x)$, and our coefficients would be $\frac{\sqrt{3}-3}{6}$. As a result, we have approximately $\log_{2+\sqrt{3}} \frac{6\cdot 2^{75}}{3-\sqrt{3}}$ which, rounded down, rounds to $\boxed{40}$. □

________ 36. **[15]** 30 points are located on a circle. How many ways are there to draw any number of line segments between the points such that none of the line segments overlap and none of the points are on more than one line segment? (It is possible to draw no line segments). If your estimate is E and the answer is A, your score for this problem will be

$$\max\left(0, \left\lceil 15 - \ln \frac{A}{E} \right\rceil\right)$$

Proposed by Powell Zhang, Richard Chen, and Brian Yu

Solution. $\boxed{1697385471211}$

Can be done through a program.

```
import java.math.BigInteger;
class Main {
public static void main(String[] args) {
BigInteger[] estimation = new BigInteger[31];
estimation[0] = new BigInteger("1");
estimation[1] = new BigInteger("1");
for(int i = 2; i < 31; i++) {
int a = i-2;
int b = 0;
BigInteger sum = new BigInteger("0");
```

```
while(b < i-1) {
BigInteger add = estimation[a].multiply(estimation[b]);
a–;
b++;
sum = sum.add(add); }
estimation[i] = sum.add(estimation[i-1]); }
for(int i = 0; i < 31; i++)
System.out.println(i + " " + estimation[i]); }
}
```

□