1. **[10]** Triangle LMT has $\overline{MA}$ as an altitude. Given that $MA = 16$, $MT = 20$, and $LT = 25$, find the length of the altitude from L to $\overline{MT}$.

   Proposed by Kevin Zhao

   Solution. $\boxed{20}$

   Let B be the foot of the angle bisector of MT. Notice that $\triangle MAT \sim \triangle LBT$ meaning that

   $$\frac{MA}{MT} = \frac{LB}{LT} \implies \frac{16}{20} = \frac{LB}{25} \implies LB = \boxed{20}.$$

   □

2. **[10]** The function $f(x)$ has the property that $f(x) = -\frac{1}{f(x-1)}$. Given that $f(0) = -\frac{1}{21}$, find the value of $f(2021)$.

   Proposed by Ada Tsui

   Solution. $\boxed{21}$

   Note that the function repeats $-\frac{1}{21}, 21, -\frac{1}{21}, 21, \dots$ starting from $f(0)$. That is, all even values of x yields $-\frac{1}{21}$ and all odd values of x yields 21. As 2021 is odd, $f(2021) = 21$. □

3. **[10]** Find the greatest possible sum of integers a and b such that $\frac{2021!}{20^a \cdot 21^b}$ is a positive integer.

   Proposed by Aidan Duncan

   Solution. $\boxed{837}$

   2021! has 2013 factors of 2, 503 factors of 5, and 334 factors of 7, so $a + b$ is $503 + 334 = \boxed{837}$. □

4. **[10]** Five members of the Lexington Math Team are sitting around a table. Each flips a fair coin. Given that the probability that three consecutive members flip heads is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

   Proposed by Alex Li

   Solution. $\boxed{43}$

   Casework by the number of heads gives $\frac{11}{32}$, which gives $\boxed{43}$. □

5. **[15]** In rectangle $ABCD$, points E and F are on sides $\overline{BC}$ and $\overline{AD}$, respectively. Two congruent semicircles are drawn with centers E and F such that they both lie entirely on or inside the rectangle, the semicircle with center E passes through C, and the semicircle with center F passes through A. Given that $AB = 8$, $CE = 5$, and the semicircles are tangent, find the length BC.

   Proposed by Ada Tsui

   Solution. $\boxed{16}$

   Let $BC = x$ and G be a point on BC such that FG is perpendicular to BC. Then $FG = AB = 8$ and $BG = CE = 5$, so $EG = x - 10$.

   Connect points E, F, G such that the right triangle EFG is formed. Then EF is twice the shared radii of the semicircles, $2CE = 10$.

   As EFG is a right triangle with the values of EF and FG, the value of EG can be found with the Pythagorean Theorem. That is, $EG = 6$. Of course, EG is also $x - 10$, so $x - 10 = 6$ and $x = 16$.

   Thus, $x = 16$ and $BC = 16$. □

6. **[15]** Given that the expected amount of 1s in a randomly selected 2021-digit number is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

   Proposed by Hannah Shen

Solution. $\boxed{1828}$

We can consider the sum of the probability that each individual digit is 1. The leftmost digit can be any of the 9 integers from 1 to 9 inclusive, so we expect the digit to be 1 with a $\frac{1}{9}$ chance. Each of the 2020 remaining digits can be any of the 10 integers from 0 to 9 inclusive, so we expect each of these to be 1 with a $\frac{1}{10}$ chance. Our final answer is $\frac{1}{9}+2020*\frac{1}{10}=\frac{1819}{9} \implies \boxed{1828}$. □

7. **[15]** A geometric sequence consists of 11 terms. The arithmetic mean of the first 6 terms is 63, and the arithmetic mean of the last 6 terms is 2016. Find the 7th term in the sequence.

   Proposed by Powell Zhang

   Solution. $\boxed{384}$

   $\frac{2016}{63}=32=2^5$. Since of the last six terms, each term is five terms later than its corresponding term in the first six terms. Thus we know that the common ratio of the geometric sequence is 2. We know that the sum of the first 6 terms is $t_1(2^6-1)=6\cdot 63=378 \implies t_1=6$ so we know the first term is 6. Thus, the 7th term is just $6\cdot 2^6=384$. □

8. **[15]** Isosceles $\triangle ABC$ has interior point O such that $AO=\sqrt{52}$, $BO=3$, and $CO=5$. Given that $\angle ABC=120°$, find the length AB.

   Proposed by Powell Zhang

   Solution. $\boxed{7}$

   Simply rotate O 120° counterclockwise to O'. The length of $\overline{OO'}$ will then be $3\sqrt{3}$ by law of cosines, making $\triangle OO'A$ a right triangle. From here, we get that $\angle BO'C$ is also 120° and thus $AB=7$ by Law of Cosines. □

9. **[20]** Find the sum of all positive integers n such that $7<n<100$ and 1573_n has 6 factors when written in base 10.

   Proposed by Aidan Duncan

   Solution. $\boxed{222}$

   Note that $1573_n=n^3+5n^2+7n+3=(n+1)^2(n+3)$. If both $n+1$ and $n+3$ are prime, we would have $(2+1)(1+1)=6$ factors. Otherwise, if one of $n+1$ or $n+3$ is not prime, we will end up having too many factors. Thus, we only need count those n where $n+1, n+3$ are prime. After we count everything, it ends up being $10+16+28+40+58+70=\boxed{222}$. □

10. **[20]** Pieck the Frog hops on Pascal's Triangle, where she starts at the number 1 at the top. In a hop, Pieck can hop to one of the two numbers directly below the number she is currently on with equal probability. Given that the expected value of the number she is on after 7 hops is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

    Proposed by Steven Yu

    Solution. $\boxed{445}$

    We note that on the row that the frog will end up on, we will have $\binom{7}{0},\binom{7}{1},\ldots,\binom{7}{7}$ be the possible values. To get to $\binom{7}{0}$, we need to go left 7 times and right 0 times, so we have $\binom{7}{7}$ ways to get to this value. For $\binom{7}{1}$, we need to go left 6 times and right 1 time, so we have $\binom{7}{6}$ ways to get to this value. This continues on, so that for $\binom{7}{k}$ we have $\binom{7}{7-k}$ ways to get to this value. Thus, since we have 2^7 paths we could traverse, our final expected value is

    $$\frac{\binom{7}{0}\cdot\binom{7}{7}+\binom{7}{1}\cdot\binom{7}{6}+\cdots+\binom{7}{7}\cdot\binom{7}{0}}{2^7}=\frac{\binom{14}{7}}{2^7}=\frac{429}{16},$$

    where we used Vandermonde's identity to compute our sum of products of binomials. Thus, our answer is $m+n=429+16=\boxed{445}$. □

11. **[20]** In $\triangle ABC$ with $\angle BAC=60°$ and circumcircle ω, the angle bisector of $\angle BAC$ intersects side $\overline{BC}$ at point D, and line AD is extended past D to a point A'. Let points E and F be the feet of the perpendiculars of A' onto lines AB and AC, respectively. Suppose that ω is tangent to line EF at a point P between E and F such that $\frac{EP}{FP}=\frac{1}{2}$. Given that $EF=6$, the area of $\triangle ABC$ can be written as $\frac{m\sqrt{n}}{p}$, where m and p are relatively prime positive integers, and n is a positive integer not divisible by the square of any prime. Find $m+n+p$.

    Proposed by Taiki Aiba

Solution. $\boxed{52}$

Note that quadrilateral $AEA'F$ is two $30-60-90$ right triangles AEA' and AFA' glued together by their hypotenuses $\overline{AA'}$. Thus, we have that $\triangle AEF$ is equilateral. Next, we will use Power of a Point with $\triangle AEF$ and ω. We see that $EP = 2$ and $FP = 4$. Note that line AE intersects ω at B and A, and line AF intersects ω at C and A. We have the equations

$$2^2 = BE(6) \quad \text{and} \quad 4^2 = CF(6).$$

Note that $AB = AE - BE = 6 - BE$ and $AC = AF - CF = 6 - CF$, so we get

$$4 = (6 - AB)6 \quad \text{and} \quad 16 = (6 - AC)6.$$

Solving gives us $AB = \frac{16}{3}$ and $AC = \frac{10}{3}$. Finally, we have that the area of $\triangle ABC$ is

$$\frac{16}{3} \cdot \frac{10}{3} \cdot \sin(60^\circ) \cdot \frac{1}{2} = \frac{40\sqrt{3}}{9},$$

for an answer of $40 + 3 + 9 = \boxed{52}$. □

12. **[20]** There are 23 balls on a table, all of which are either red or blue, such that the probability that there are n red balls and $23 - n$ blue balls on the table ($1 \le n \le 22$) is proportional to n. (e.g. the probability that there are 2 red balls and 21 blue balls is twice the probability that there are 1 red ball and 22 blue balls.) Given that the probability that the red balls and blue balls can be arranged in a line such that there is a blue ball on each end, no two red balls are next to each other, and an equal number of blue balls can be placed between each pair of adjacent red balls is $\frac{a}{b}$, where a and b are relatively prime positive integers, find $a + b$.
Note: There can be any nonzero number of consecutive blue balls at the ends of the line.

Proposed by Ada Tsui

Solution. $\boxed{29}$

Let the number of blue balls be denoted by b. We have the requirement, first, that there be at least two blue balls to fill in the ends. Next, we just need that

$$b - 2 \ge k(23 - b) - k \implies (22 - b)(k + 1) \le 20 \implies b \ge 12,$$

so that we can insert k blue balls in between every pair of adjacent red balls, and then put the leftover blue balls in random places, just not between any red balls. Now, our probability is

$$\frac{1 + 2 + 3 + \cdots + 11}{1 + 2 + 3 + \cdots + 22} = \frac{6}{23} \implies \boxed{29}.$$

□

13. **[25]** In a round-robin tournament, where any two players play each other exactly once, the fact holds that among every three students A, B, and C, one of the students beats the other two. Given that there are six players in the tournament and Aidan beats Zach but loses to Andrew, find how many ways there are for the tournament to play out.
Note: The order in which the matches take place does not matter.

Proposed by Kevin Zhao

Solution. $\boxed{120}$

Note that this means if A beats B and B beats C, then A beats C. As a result, we have that we can order the six players in ability such that a higher ability person beats a lower ability person. Now, we have players X, Y, Z, Aidan, Zach, and Andrew. We now see that Aidan, Zach, and Andrew are in a specific order and X, Y, Z aren't so there are 6 ways to order X, Y and Z. We have $\binom{6}{3} = 20$ ways to choose which three people in ability order are X, Y, and Z, so our final answer is $6 \cdot 20 = \boxed{120}$. □

14. **[25]** Alex, Bob, and Chris are driving cars down a road at distinct constant rates. All people are driving a positive integer number of miles per hour. All of their cars are 15 feet long. It takes Alex 1 second longer to completely pass Chris than it takes Bob to completely pass Chris. The passing time is defined as the time where their cars overlap. Find the smallest possible sum of their speeds, in miles per hour.

Proposed by Sammy Charney

Solution. $\boxed{171}$

Let their speeds be a, b, and c miles per hour, respectively, and let $x = a-c$ and $y = b-c$. One mile per hour is equivalent to $\frac{5280}{3600} = \frac{22}{15}$ feet per second. In order to pass a car, one must travel the sum of their lengths, or 30 feet. The time it takes to travel 30 feet is $\frac{30}{\frac{22s}{15}}$ when travelling s miles per hour. This gives us the equation $\frac{450}{22x} = \frac{450}{22y} + 1$. Simplifying we get $450y = 450x + 22xy$, so $(11x-225)(11y+225) = -225^2 = -50625$. As $11x-225$ must be negative, we see $225-11x$ must be a divisor of 225^2 smaller than 225 equivalent to 5 mod 11. Checking divisors, we see only 5 and 27 work. $225-11x = 5$ yields $x = 20$ and $y = 900$. We want to minimize the sum of the speeds so we let $c = 1$, and thus $a = 21$ and $b = 901$, giving us a sum of 923. $225-11x = 27$ yields $x = 18$ and $y = 150$. Letting $c = 1$ again, we have $a = 19$ and $b = 151$, giving us a minimal sum of $\boxed{171}$. □

15. **[25]** Andy and Eddie play a game in which they continuously flip a fair coin. They stop flipping when either they flip tails, heads, and tails consecutively in that order, or they flip three tails in a row. Then, if there has been an odd number of flips, Andy wins, and otherwise Eddie wins. Given that the probability that Andy wins is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

Proposed by Andrew Zhao & Zachary Perry

Solution. $\boxed{39}$

Let H_o be the probability that Andy wins after an "odd heads reset", where there has been a heads flipped on an odd number of flips, and it acts as a reset, so that nothing before that will affect the future probabilities. Let H_e, T_o, T_e denote similar values.

Now, we can create a system of equations for this:

$$H_o = \frac{1}{2}\cdot T_e + \frac{1}{2}\cdot H_e$$
$$H_e = \frac{1}{2}\cdot T_o + \frac{1}{2}\cdot H_o$$
$$T_o = \frac{1}{2}\cdot\left(\frac{1}{2}\cdot 1 + \frac{1}{2}H_o\right) + \frac{1}{2}\cdot\left(\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\left(\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot H_e\right)\right) \implies T_o = \frac{1}{2} + \frac{1}{4}H_o + \frac{1}{8}H_e$$
$$T_e = \frac{1}{2}\cdot\left(\frac{1}{2}\cdot 0 + \frac{1}{2}H_e\right) + \frac{1}{2}\cdot\left(\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot\left(\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot H_0\right)\right) \implies T_e = \frac{1}{8} + \frac{1}{4}H_e + \frac{1}{8}H_o$$

Solving this system of equations, we find that $H_o = \frac{11}{25}, T_o = \frac{17}{25}$. Thus, the probability that Andy takes home the champions' belt is $\frac{1}{2}(H_o + T_o) = \frac{14}{25} \implies \boxed{39}$. (Solution by Richard Chen) □

16. **[25]** Find the number of ordered pairs (a, b) of positive integers less than or equal to 20 such that

$$\gcd(a,b) > 1 \quad \text{and} \quad \frac{1}{\gcd(a,b)} + \frac{a+b}{\operatorname{lcm}(a,b)} \geq 1.$$

Proposed by Zachary Perry

Solution. $\boxed{95}$

For ease of writing, I will refer to $\gcd(a, b)$ as g. Let the numbers A and B be the positive integers such that $Ag = a$ and $Bg = b$. Clearly A and B are relatively prime, otherwise g would not be maximized. An important fact is that $g\cdot\operatorname{lcm}(a,b) = ab$. Thus, we can write the condition as

$$\frac{1}{g} + \frac{ag+bg}{ab} = \frac{1}{g} + \frac{g}{a} + \frac{g}{b} = \frac{1}{g} + \frac{1}{A} + \frac{1}{B} \geq 1$$

.

Now, we use casework on the value of g. We make a table (excluding permutations in the possible pairs column):

g	Possible pairs (A, B)	Total
2	(1,[1,2,3,... 10]), (2,[3,5,7,9]), (3,[4,5])	31
3	(1,[1,2,3,... 6]), (2,[3,5])	15
4	(1,[1,2,3,4,5]), (2,3)	11
5	(1,[1,2,3,4]), (2,3)	9
6	(1,[1,2,3]), (2,3)	7

For the remaining g, from $7-10$, $(1,1)$, $(1,2)$, and $(2,1)$ all work, and from $11-20$, only $(1,1)$ works. This gives a total sum of $31+15+11+9+7+4\cdot 3+10 = \boxed{95}$.

Verified by code (Thanks Aidan and Alex!). $\square$

17. **[30]** Given that the value of

$$\sum_{k=1}^{2021} \frac{1}{1^2+2^2+3^2+\cdots+k^2} + \sum_{k=1}^{1010} \frac{6}{2k^2-k} + \sum_{k=1011}^{2021} \frac{24}{2k+1}$$

can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

Proposed by Aidan Duncan

Solution. $\boxed{12938440}$

First,

$$\sum_{k=1}^{2021} \frac{1}{1^2+2^2+3^2+\cdots+k^2} = \sum_{k=1}^{2021} \frac{6}{k(k+1)(2k+1)}.$$

Then, note that

$$\frac{1}{k(k+1)(2k+1)} = \frac{1}{k} + \frac{1}{k+1} - \frac{4}{2k+1} \implies \sum_{k=1}^{2021} \frac{6}{k(k+1)(2k+1)} = 6\sum_{k=1}^{2021} \frac{1}{k} + \frac{1}{k+1} - \frac{4}{2k+1}.$$

Simplifying, we get

$$6\sum_{k=1}^{2021} \frac{1}{k} + \frac{1}{k+1} - \frac{4}{2k+1} = 6\left(1 + \frac{1}{2022} + 2\sum_{k=2}^{2021} \frac{1}{k}\cdot -1^k + \sum_{k=1011}^{2021} \frac{-4}{2k+1}\right),$$

so adding the third summation removes the last term and leaves us with

$$6 + \frac{6}{2022} + 12\sum_{k=2}^{2021} \frac{1}{k}\cdot -1^k.$$

Now, in the second summation we have that $\frac{6}{2k^2-k} = \frac{12}{2k(2k-1)}$, so our total is

$$6 + \frac{6}{2022} + 12\sum_{k=2}^{2021} \frac{1}{k}\cdot -1^k + 12\sum_{k=1}^{1010} \frac{1}{2k(2k-1)}.$$

Then, we get that

$$12\sum_{k=2}^{2021} \frac{1}{k}\cdot -1^k = 12\left(\left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{4}-\frac{1}{5}\right) + \cdots + \left(\frac{1}{2020}-\frac{1}{2021}\right)\right) = 12\sum_{k=1}^{1010} \frac{1}{2k(2k+1)}.$$

Combining this with $12\sum_{k=1}^{1010} \frac{1}{2k(2k-1)}$ gives

$$12\sum_{k=1}^{2020} \frac{1}{k(k+1)} = 12\left(\left(\frac{1}{1}-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \cdots + \left(\frac{1}{2020}-\frac{1}{2021}\right)\right) = 12 - \frac{12}{2021}.$$

Adding this back to $6+\frac{6}{2022}$ gives $18+\frac{6}{2022}-\frac{12}{2021}$ as our answer, which can be simplified down to $\frac{12257363}{681077} \implies \boxed{12938440}$. $\square$

18. **[30]** Points X and Y are on a parabola of the form $y = \frac{x^2}{a^2}$ and A is the point $(x,y) = (0,a)$. Assume XY passes through A and hits the line $y = -a$ at a point B. Let ω be the circle passing through $(0,-a)$, A, and B. A point P is chosen on ω such that $PA = 8$. Given that X is between A and B, $AX = 2$, and $XB = 10$, find $PX \cdot PY$.

Proposed by Kevin Zhao

Solution. $\boxed{70}$

Note that A is the focus of the parabola and B is on the directrix of the parabola. As a result, we see that $(AB;XY) = -1$. Now, note that $(0,-a)$ is on the circle with diameter AB and so $\angle APB = 90°$. Because $(AB;XY) = -1$, then we see that $\angle XPA = \angle YPA$ and so

$$XP \cdot YP = AP^2 + XA \cdot YA = 64 + 2 \cdot YA$$

and now we see that we can use ratios to find AY. We see that $AY \cdot BX = XA \cdot BY$ because of the harmonic pencil, and so $AY \cdot 10 = 2 \cdot (AY + 12)$ and so $AY = 3$. So, $XP \cdot YP = 64 + 2 \cdot 3 = \boxed{70}$. □

19. **[30]** Let S be the sum of all possible values of $a \cdot c$ such that

$$a^3 + 3ab^2 - 72ab + 432a = 4c^3$$

if a, b, and c are positive integers, $a + b > 11$, $a > b - 13$, and $c \le 1000$. Find the sum of all distinct prime factors of S.

Proposed by Kevin Zhao

Solution. $\boxed{90}$

Multiply both sides of the Diophantine Equation by 2, which gets us that

$$2(a^3 + 3ab^2 - 72ab + 432a) = 2a^3 + 6ab^2 - 144ab + 864a = 8c^3.$$

Now, note that the left side factors and

$$2a^3 + 6ab^2 - 144ab + 864a = (a + b - 12)^3 + (a - b + 12)^3,$$

and similarly, we also have that $8c^3 = (2c)^3$. Thus, we see that

$$(a + b - 12)^3 + (a - b + 12)^3 = (2c)^3$$

and we use Fermat's Last Theorem, which shows us that if $x^3 + y^3 = z^3$ for any nonnegative integers x, y, and z, then either $x = 0$ or $y = 0$. Because $a + b > 1112$, then $(a + b - 12)^3 \ge 0$, and because $a > b - 13$, then $a - b + 12 \ge 0$. As a result, because $c > 0$, we can apply Fermat's Last Theorem which shows that $a + b = 12$ or $a = b - 12$. Plugging in values shows that if $a + b = 12$ then we have solutions in form of $0^3 + (a - (12 - a) + 12)^3 = (2c)^3 \to a = c$ where $1 \le a \le 11$, and if $b = a + 12$ then $(a + (a + 12) - 12)^3 + 0^3 = (2c)^3 \to a = c$ where $a \ge 1$. Hence, we see that $a = c$ and $a \ge 1$. Hence, our sum would be

$$1^2 + 2^2 + \ldots + 1000^2 = \frac{1000 \cdot 1001 \cdot 2001}{6} = 500 \cdot 1001 \cdot 667.$$

So, we note that the sum of the union of prime divisors of $500 = 2^2 \cdot 5^3$, $1001 = 7 \cdot 11 \cdot 13$, and $667 = 23 \cdot 29$ would be $2 + 5 + 7 + 11 + 13 + 23 + 29 = \boxed{90}$. □

20. **[30]** Let Ω be a circle with center O. Let ω_1 and ω_2 be circles with centers O_1 and O_2, respectively, internally tangent to Ω at points A and B, respectively, such that O_1 is on $\overline{OA}$, and O_2 is on $\overline{OB}$ and ω_1. There exists a point P on line AB such that P is on both ω_1 and ω_2. Let the external tangent of ω_1 and ω_2 on the same side of line AB as O hit ω_1 at X and ω_2 at Y, and let lines AX and BY intersect at N. Given that $O_1X = 81$ and $O_2Y = 18$, the value of $NX \cdot NA$ can be written as $a\sqrt{b} + c$, where a, b, and c are positive integers, and b is not divisible by the square of a prime. Find $a + b + c$.

Proposed by Kevin Zhao

Solution. $\boxed{4586}$

Let Q be the other point besides P where ω_1 and ω_2 meet. First, note that taking a homothety centered at B from P to A takes O_2 to O, and taking a homothety centered at A from P to B takes O_2 to O. Hence, $\angle BO_2P = \angle BOA = \angle PO_1A$ and so OO_1PO_2 is a parallelogram. Since O_2 is on ω_1, then $O_2O_1 = O_2O$.

Note that letting N' be the point where the homothety centered at B takes Y to, then there must be a line l tangent to Ω at N' parallel to XY. Let the homothety centered at A take X to a point N'', and we see that because there must be a line l tangent to Ω at N'' parallel to XY, then $N = N''$. $N' = N''$ is also on Ω, obviously, because of the homothety. So, $N = N' = N''$ too because that's the point where the two lines meet. We also see that Archimedes's Lemma now states that the power of N with respect to ω_1 and ω_2 is the same, hence P, Q, and N are collinear, and $NP \cdot NQ = NX \cdot NA = NY \cdot NB$ (An alternate to Archimedes's Lemma would be inverting about a circle with center N and radius $\sqrt{NX \cdot NA}$; this gets that X maps to A and Y maps to B because XY maps to Ω, and so $ABYX$ is cyclic. By Radical Axis Theorem, we see that PQ, AX, and BY concur at N).

Let $OO_2 = O_1P = O_1A = r_1$ and $OO_1 = O_2P = O_2B = r_2$. Note that now, $O_1O_2 = O_1P = r_1$.

As a result, we see that $\cos(\angle AOB) = \frac{r_2}{2r_1}$ and so

$$AB = (r_1 + r_2)\sqrt{2 - 2\cos(\angle AOB)} = (r_1 + r_2)\sqrt{2 - \frac{r_2}{r_1}}.$$

Plugging in $r_1 = 81$ and $r_2 = 18$ gets us that we have $AB = 132$. Since

$$\frac{AP}{PB} = \frac{AO_1}{O_1O} = \frac{r_1}{r_2} = \frac{81}{18} = \frac{9}{2}$$

then we see that $AP = \frac{9}{11} \cdot AB = 108$ and $BP = \frac{2}{11} \cdot 132 = 24$.

We note that $\angle O_1OO_2 = \angle O_2PO_1 = \angle O_1QO_2$ which means that OQ and O_1O_2 are parallel and so since O_1O_2 and PQ are parallel, then OQ is parallel to PQ. Let ray QP hit Ω at M. Then, we see that OQ bisects MN because $OM = ON$ and $\angle OQM = \angle OQN = 90°$.

So, we see that

$$MP \cdot NP = (MQ - PQ)(NQ + QP) = (NQ + QP)(NQ - QP)$$

which is just

$$NQ^2 - QP^2 = ON^2 - OQ^2 - QP^2$$

due to the Pythagorean Theorem and the fact that $MQ = NQ$. This shows that $ON^2 - OQ^2 - QP^2 = AP \cdot BP$. To find OQ, we use Ptolemy's Theorem on cyclic quadrilateral O_1O_2QO and get that $OQ = \frac{r_1^2 - r_2^2}{r_1} = 81 - 4 = 77$, and so because $ON = r_1 + r_2 = 99$, we have that

$$99^2 - 77^2 - QP^2 = 108 \cdot 24 \to QP^2 = 1280.$$

Hence, because $NQ^2 = ON^2 - OQ^2 = 99^2 - 77^2$, then $NQ = 44\sqrt{2}$ and we also see from before that now, $PQ = 16\sqrt{5}$. Finally, we apply Power of a Point, and see that

$$NX \cdot NA = NQ \cdot NP = 44\sqrt{2} \cdot (44\sqrt{2} + 16\sqrt{5})$$

$$= 704\sqrt{10} + 3872 \to 704 + 10 + 3872 = \boxed{4586}$$

which is our final answer, and we are done. □

Haikus

A Haiku is a Japanese poem of seventeen syllables, in three lines of five, seven, and five.

21. **[15]** In how many ways
Can you add three integers
Summing seventeen?

Order matters here.
For example, eight, three, six
Is not eight, six, three.

All nonnegative,
Do not need to be distinct.
What is your answer?

Proposed by Derek Gao

Solution. $\boxed{171}$

Stars and bars gives $\binom{17+3-1}{3-1} = \binom{19}{2} = \boxed{171}$. □

22. **[20]** Ada has been told
To write down five haikus plus
Two more every hour.

Such that she needs to
Write down five in the first hour
Seven, nine, so on.

Ada has so far
Forty haikus and writes down
Seven every hour.

At which hour after
She begins will she not have
Enough haikus done?

Proposed by Ada Tsui

Solution. $\boxed{9}$

We want to solve the inequality

$$40 + 7h < \sum_{i=0}^{h-1}(5+2i) \implies 40 + 7h < 5(h) + 2 \cdot \frac{(h-1)h}{2} \implies 40 < (h-3)h \implies h = \boxed{9}$$

is the earliest hour where this inequality holds. □

23. **[20]** A group of haikus
Some have one syllable less
Sixteen in total.

The group of haikus
Some have one syllable more
Eighteen in total.

What is the largest
Total count of syllables
That the group can't have?

(For instance, a group
Sixteen, seventeen, eighteen
Fifty-one total.)

(Also, you can have
No sixteen, no eighteen
Syllable haikus)

Proposed by Jeff Lin

Solution. $\boxed{127}$

Eighteen equals one
(But modulo seventeen)
Sixteen = minus one

After one-two-eight
Eight times eighteen, or sixteen
gets us ±eight mod 17

Then $\boxed{\text{one-two-seven}}$
We try and do not succeed
To get this number

□

24. **[25]** Using the four words
"Hi", "hey", "hello", and "haiku",
How many haikus

Can somebody make?
(Repetition is allowed,
Order does matter.)

Proposed by Jeff Lin

Solution. $\boxed{12902400}$

Two 2s and two ones
And the fives have three cases
Two two one has three

Multiply by eight
And that gives us twenty four
Two one one one - four

Times with sixteen
And then we get sixty-four
Five ones has one way

So that's thirty-two
Adding gives us one-twenty
The line of seven

Has four cases now
Two-two-two-one has four ways
Multiply sixteen

Gives sixty-four ways
Two-two-one-one-one is ten
And times thirty-two

Three-hundred twenty
Two and five ones is six
And times sixty-four

Gives three-eighty-four.
Seven ones, one-twenty eight
Added up we get

Eight-ninety-six ways
One-twenty squared times that is

twelve million,

nine-o-two thousand
four hundred, and that is it.
That was way too long. □

Chandler the Octopus

25. **[15]** Chandler the Octopus is making a concoction to create the perfect ink. He adds 1.2 grams of melanin, 4.2 grams of enzymes, and 6.6 grams of polysaccharides. But Chandler accidentally added n grams of an extra ingredient to the concoction, Chemical X, to create glue. Given that Chemical X contains none of the three aforementioned ingredients, and the percentages of melanin, enzymes, and polysaccharides in the final concoction are all integers, find the sum of all possible positive integer values of n.

    Proposed by Taiki Aiba

    Solution. $\boxed{77}$

    The problem statement is basically asking for the sum of all n such that $\frac{120}{12+n}$, $\frac{420}{12+n}$, and $\frac{660}{12+n}$ are all integers. This can be found by taking the number of grams of an ingredient, dividing it by the total number of grams in the final concoction, which is $1.2+4.2+6.6+n=12+n$, and multiplying the result by 100 to account for it being a percentage. Note that a value of n satisfies the conditions if and only if $12+n$ is a divisor of all three of 120, 420, and 660. Note that $120=2^3\cdot3\cdot5$, $420=2^2\cdot3\cdot5\cdot7$, and $660=2^2\cdot3\cdot5\cdot11$, so the greatest common divisor of these three numbers is $2^2\cdot3\cdot5=60$. Also, note that every divisor of 60 will also be a divisor of all twhree of 120, 420, and 660. We find that the divisors of 60 are 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, and 60. Of these divisors, only 15, 20, 30, and 60 are greater than 12. Since these divisors are values of $12+n$, we find that the values of n are 3, 8, 18, and 48, where we subtracted 12 from each of the four divisors. Thus, our final answer is $3+8+18+48=\boxed{77}$. □

26. **[20]** Chandler the Octopus along with his friends Maisy the Bear and Jeff the Frog are solving LMT problems. It takes Maisy 3 minutes to solve a problem, Chandler 4 minutes to solve a problem and Jeff 5 minutes to solve a problem. They start at 12:00 pm, and Chandler has a dentist appointment from 12:10 pm to 12:30, after which he comes back and continues solving LMT problems. The time it will take for them to finish solving 50 LMT problems, in hours, is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$. **Note: they may collaborate on problems.**

    Proposed by Aditya Rao

    Solution. $\boxed{102}$

    When all 3 work together, they work at a rate of $\frac{1}{3}+\frac{1}{4}+\frac{1}{5}=\frac{47}{60}$ problems per minute. When Chandler is absent, they work at a rate of $\frac{1}{3}+\frac{1}{5}=\frac{8}{15}$ problems per minute. Thus, we have $(x-20)\cdot\frac{47}{60}+20\cdot\frac{8}{15}=50 \implies x=\frac{118\cdot20}{47}$, so the total amount of time we use is $\frac{118\cdot20}{47}+20=\frac{165\cdot20}{47}$ minutes, or $\frac{165\cdot20/60}{47}=\frac{55}{47}$ hours. Thus, our answer is $m+n=55+47=\boxed{102}$. □

27. **[25]** Chandler the Octopus is at a tentacle party!

    At this party, there is 1 creature with 2 tentacles, 2 creatures with 3 tentacles, 3 creatures with 4 tentacles, all the way up to 14 creatures with 15 tentacles. Each tentacle is distinguishable from all other tentacles. For some $2\le m<n\le15$, a creature with m tentacles "meets" a creature with n tentacles; "meeting" another creature consists of shaking exactly 1 tentacle with each other. Find the number of ways there are to pick distinct $m<n$ between 2 and 15, inclusive, and then to pick a creature with m tentacles to "meet" a selected creature with n tentacles.

    Proposed by Armaan Tipirneni, Richard Chen, and Denise the Octopus

    Solution. $\boxed{551824}$

    We find that the expression is equal to $\sum_{1\le m<n\le15} m(m-1)n(n-1)$. This is equal to:

    $$\frac{1}{2}\left(\left(\sum_{m=2}^{15} m(m-1)\right)^2-\left(\sum_{m=2}^{15}(m(m-1))^2\right)\right)=\frac{1}{2}\left(4\binom{16}{3}^2-\sum_{m=1}^{15}\left(m^4-2m^3+m^2\right)\right)=\boxed{551824}.$$

    Note that the sum of m^4, m^3, m^2 can all be calculated by what is called Faulhaber's formula (the sum of $1^k+2^k+\cdots+n^k$ can be computed with a $k+1$ degree polynomial). □

Card Games

28. **[15]** Addison and Emerson are playing a card game with three rounds. Addison has the cards 1, 3, and 5, and Emerson has the cards 2, 4, and 6. In advance of the game, both designate each one of their cards to be played for either round one, two, or three. Cards cannot be played for multiple rounds. In each round, both show each other their designated card for that round, and the person with the higher-numbered card wins the round. The person who wins the most rounds wins the game. Let $\frac{m}{n}$ be the probability that Emerson wins, where m and n are relatively prime positive integers. Find $m+n$.

Proposed by Ada Tsui

Solution. $\boxed{11}$

Let Addison designate her cards in the order $1,3,5$.

The total number of ways Emerson can designate her cards is $3! = 6$.

Addison must win two or three of the rounds to win the game.

In the first round, all of Emerson's cards are higher than Addison's card, 1, so Emerson automatically wins this round and Addison must win the remaining two rounds to win the game.

In the second round, only Emerson's card of 2 is lower than Addison's card, 3, so Emerson must play a 2 for Addison to win this round and have a chance to win the game.

In the third round, Emerson's cards of 2 and 4 are lower than Addison's card, 5, but Emerson already used the 2 in the second round, so Emerson must play a 4 for Addison to win this round and the game.

Thus, the only way for Addison to win is for Emerson to designate her cards in the order $6,2,4$. Of course, we are looking for the number of ways for Emerson to win, and that is $6-1=5$.

The probability Emerson wins is $\frac{5}{6}$, so $m=5, n=6$, and $m+n=\boxed{11}$. □

29. **[20]** In a group of 6 people playing the card game Tractor, all 54 cards from 3 decks are dealt evenly to all the players at random. Each deck is dealt individually. Let the probability that no one has at least two of the same card be X. Find the largest integer n such that the nth root of X is rational.

Proposed by Sammy Charney

Solution. $\boxed{}$

Due to the problem having infinitely many solutions, all teams who inputted answers received points. □

30. **[25]** Ryan Murphy is playing poker. He is dealt a hand of 5 cards. Given that the probability that he has a straight hand (the ranks are all consecutive; e.g. $3,4,5,6,7$ or $9,10,J,Q,K$) or 3 of a kind (at least 3 cards of the same rank; e.g. $5,5,5,7,7$ or $5,5,5,7,K$) is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

Proposed by Aditya Rao

Solution. $\boxed{55572}$

We compute the number of possibilities for a straight hand, and then the number of possibilities for a 3 of a kind.

For a straight hand, our numerical values can be $\{1,2,3,4,5\},\{2,3,4,5,6\},\dots,\{9,10,11,12,13\}$. Thus, we have 9 possibilities here. For each of those numerical values, we have 4 suits from which we can choose. Thus, the total number of ways we can obtain a straight hand is $9\cdot 4^5$.

For a 3 of a kind, we have two different possibilities: that we get exactly 3 of a kind and 2 other cards, or that we get 4 of a kind and 1 other card. For exactly 3 of a kind and 2 other cards, we have 13 ways to choose which number has the 3 of a kind, $\binom{4}{3}$ ways to choose the 3 suits for that 3 of a kind, and then $\binom{48}{2}$ ways to choose our 2 other cards, so we have $13\cdot\binom{4}{3}\cdot\binom{48}{2}$ ways in this first case. In the second case, we have 13 ways to choose which number has 4 of a kind, and then $\binom{48}{1}$ to choose our 1 remaining card, so we have $13\cdot\binom{48}{1}$.

In total, we have $\binom{52}{5}$ total possibilities when selecting our hand. So, the probability that we have a straight hand or 3 of a kind is

$$\frac{9 \cdot 4^5 + 13 \cdot \binom{4}{3} \cdot \binom{48}{2} + 13 \cdot \binom{48}{1}}{\binom{52}{5}} = \frac{1427}{54145} \implies \boxed{55572}.$$

$\square$