LMT Spring Online: Division B

April 30th – May 7th, 2021

Contest Instructions

Contest Window

The Team Round consists of 30 short answer problems, including 10 theme problems. All answers are non-negative integers. The problems will be made available on the homepage of the LMT website on **Friday, April 30th, at 3:30 pm EDT**. Teams will have until **Friday, May 7th, at 11:59 pm EDT** to submit their answers using the link provided by email.

Contest Rules

With the exception of **standard four-function calculators**, computational aids including but not limited to scientific and graphing calculators, computer programs, and software such as Geogebra, Mathematica, and WolframAlpha, are **not** allowed. Communication of any form between students on different teams is similarly prohibited, and any team caught either giving or receiving an unfair advantage over other competitors will be disqualified. What constitutes cheating will be up to the final discretion of the competition organizers, who reserve the right to disqualify any team suspected of violating these rules.

Submitting Answers and Editing Team Information

During registration, your captain will be emailed a link to your team's homepage. This is where you will be able to update team information and answer submissions. We recommend that your captain distribute this link to the rest of the team so that the entire team has access. Once on your team's homepage, to enter or edit team answers, click the link next to "Submission Link:". Team name, team member names, and grades may be edited through the homepage as well. Remember, team member names must be the real names of the people on your team, and the team name must be appropriate.

Errata

If you believe there to be an error in one of the questions, email us at lmt@lhsmath.org with "Clarification" as the subject. Clarifications for problems will be updated on the LMT homepage, if necessary.

Scoring

The score of your team is the sum of the point values of the problems you answered correctly. **Note that the theme problems deviate from the general trend in point values.** Ties will not be broken. Results will be posted shortly after the competition, and the top teams will be recognized.

1. **[10]** Given that the expression
$$\frac{20^{21}}{20^{20}} + \frac{20^{20}}{20^{21}}$$
can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

2. **[10]** Find the greatest possible distance between any two points inside or along the perimeter of an equilateral triangle with side length 2.

3. **[10]** Aidan rolls a pair of fair, six sided dice. Let n be the probability that the product of the two numbers at the top is prime. Given that n can be written as $\frac{a}{b}$, where a and b are relatively prime positive integers, find $a+b$.

4. **[10]** Set S contains exactly 36 elements in the form of $2^m \cdot 5^n$ for integers $0 \le m, n \le 5$. Two distinct elements of S are randomly chosen. Given that the probability that their product is divisible by 10^7 is $\frac{a}{b}$, where a and b are relatively prime positive integers, find $a+b$.

5. **[15]** Find the number of ways there are to permute the elements of the set $\{1,2,3,4,5,6,7,8,9\}$ such that no two adjacent numbers are both even or both odd.

6. **[15]** Maisy is at the origin of the coordinate plane. On her first step, she moves 1 unit up. On her second step, she moves 1 unit to the right. On her third step, she moves 2 units up. On her fourth step, she moves 2 units to the right. She repeats this pattern with each odd-numbered step being 1 unit more than the previous step. Given that the point that Maisy lands on after her 21st step can be written in the form (x, y), find the value of $x+y$.

7. **[15]** Given that x and y are positive real numbers such that
$$\frac{5}{x} = \frac{y}{13} = \frac{x}{y},$$
find the value of $x^3 + y^3$.

8. **[15]** Find the number of arithmetic sequences a_1, a_2, a_3 of three nonzero integers such that the sum of the terms in the sequence is equal to the product of the terms in the sequence.

9. **[20]** Convex pentagon $PQRST$ has $PQ = TP = 5$, $QR = RS = ST = 6$, and $\angle QRS = \angle RST = 90^\circ$. Given that points U and V exist such that $RU = UV = VS = 2$, find the area of pentagon $PQUVT$.

10. **[20]** Let $f(x)$ be a function mapping real numbers to real numbers. Given that $f(f(x)) = \frac{1}{3x}$, and $f(2) = \frac{1}{9}$, find $f\left(\frac{1}{6}\right)$.

11. **[20]** In rectangle $ABCD$, points E and F are on sides $\overline{BC}$ and $\overline{AD}$, respectively. Two congruent semicircles are drawn with centers E and F such that they both lie entirely on or inside the rectangle, the semicircle with center E passes through C, and the semicircle with center F passes through A. Given that $AB = 8$, $CE = 5$, and the semicircles are tangent, find the length BC.

12. **[20]** Given that the expected amount of 1s in a randomly selected 2021-digit number is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

13. **[25]** Call a 4-digit number $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}$ *unnoticeable* if $a+c = b+d$ and $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d} + \underline{c}\,\underline{d}\,\underline{a}\,\underline{b}$ is a multiple of 7. Find the number of unnoticeable numbers.
Note: a, b, c, and d are nonzero distinct digits.

14. **[25]** In the expansion of
$$(2x+3y)^{20},$$
find the number of coefficients divisible by 144.

15. **[25]** A geometric sequence consists of 11 terms. The arithmetic mean of the first 6 terms is 63, and the arithmetic mean of the last 6 terms is 2016. Find the 7th term in the sequence.

16. **[25]** Bob plants two saplings. Each day, each sapling has a $\frac{1}{3}$ chance of instantly turning into a tree. Given that the expected number of days it takes both trees to grow is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

17. **[30]** In $\triangle ABC$ with $\angle BAC = 60^\circ$ and circumcircle ω, the angle bisector of $\angle BAC$ intersects side $\overline{BC}$ at point D, and line AD is extended past D to a point A'. Let points E and F be the feet of the perpendiculars of A' onto lines AB and AC, respectively. Suppose that ω is tangent to line EF at a point P between E and F such that $\frac{EP}{FP} = \frac{1}{2}$. Given that $EF = 6$, the area of $\triangle ABC$ can be written as $\frac{m\sqrt{n}}{p}$, where m and p are relatively prime positive integers, and n is a positive integer not divisible by the square of any prime. Find $m+n+p$.

18. **[30]** There are 23 balls on a table, all of which are either red or blue, such that the probability that there are n red balls and $23-n$ blue balls on the table $(1 \le n \le 22)$ is proportional to n. (e.g. the probability that there are 2 red balls and 21 blue balls is twice the probability that there are 1 red ball and 22 blue balls.) Given that the probability that the red balls and blue balls can be arranged in a line such that there is a blue ball on each end, no two red balls are next to each other, and an equal number of blue balls can be placed between each pair of adjacent red balls is $\frac{a}{b}$, where a and b are relatively prime positive integers, find $a+b$.
Note: There can be any number of consecutive blue balls at the ends of the line.

19. **[30]** Kevin is at the point $(19,12)$. He wants to walk to a point on the ellipse $9x^2 + 25y^2 = 8100$, and then walk to $(-24,0)$. Find the shortest length that he has to walk.

20. **[30]** Andy and Eddie play a game in which they continuously flip a fair coin. They stop flipping when either they flip tails, heads, and tails consecutively in that order, or they flip three tails in a row. Then, if there has been an odd number of flips, Andy wins, and otherwise Eddie wins. Given that the probability that Andy wins is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

Haikus

A Haiku is a Japanese poem of seventeen syllables, in three lines of five, seven, and five.

21. **[15]** Take five good haikus
Scramble their lines randomly
What are the chances

That you end up with
Five completely good haikus
(With five, seven, five)?

Your answer will be
m over n where m, n
Are numbers such that

m, n positive
Integers where gcd
Of m, n is 1.

Take this answer and
Add the numerator and
Denominator.

22. **[20]** In how many ways
Can you add three integers
Summing seventeen?

Order matters here.
For example, eight, three, six
Is not eight, six, three.

All nonnegative,
Do not need to be distinct.
What is your answer?

23. **[20]** Ada has been told
To write down five haikus plus
Two more every hour.

Such that she needs to
Write down five in the first hour
Seven, nine, so on.

Ada has so far
Forty haikus and writes down
Seven every hour.

At which hour after
She begins will she not have
Enough haikus done?

24. **[25]** A group of haikus
Some have one syllable less
Sixteen in total.

The group of haikus
Some have one syllable more
Eighteen in total.

What is the largest
Total count of syllables
That the group can't have?

(For instance, a group
Sixteen, seventeen, eighteen
Fifty-one total.)

Chandler the Octopus

25. **[15]** Chandler the Octopus is using his eight tentacles to count in binary, where holding up a tentacle signifies a 1 and keeping his tentacle down signifies a 0. Find the sum of the base-10 values of all the numbers in which Chandler is holding up exactly 3 tentacles.

26. **[20]** Chandler the Octopus is making a concoction to create the perfect ink. He adds 1.2 grams of melanin, 4.2 grams of enzymes, and 6.6 grams of polysaccharides. But Chandler accidentally added n grams of an extra ingredient to the concoction, Chemical X, to create glue. Given that Chemical X contains none of the three aforementioned ingredients, and the percentages of melanin, enzymes, and polysaccharides in the final concoction are all integers, find the sum of all possible positive integer values of n.

27. **[25]** Chandler the Octopus along with his friends Maisy the Bear and Jeff the Frog are solving LMT problems. It takes Maisy 3 minutes to solve a problem, Chandler 4 minutes to solve a problem and Jeff 5 minutes to solve a problem. They start at 12:00 pm, and Chandler has a dentist appointment from 12:10 pm to 12:30, after which he comes back and continues solving LMT problems. The time it will take for them to finish solving 50 LMT problems, in hours, is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$. **Note: they may collaborate on problems.**

Card Games

28. **[15]** Maisy and Jeff are playing a game with a deck of cards with 4 0's, 4 1's, 4 2's, all the way up to 4 9's. You cannot tell apart cards of the same number. After shuffling the deck, Maisy and Jeff each take 4 cards, make the largest 4-digit integer they can, and then compare. The person with the larger 4-digit integer wins. Jeff goes first and draws the cards $2,0,2,1$ from the deck. Find the number of hands Maisy can draw to beat that.

29. **[20]** Addison and Emerson are playing a card game with three rounds. Addison has the cards 1, 3, and 5, and Emerson has the cards 2, 4, and 6. In advance of the game, both designate each one of their cards to be played for either round one, two, or three. Cards cannot be played for multiple rounds. In each round, both show each other their designated card for that round, and the person with the higher-numbered card wins the round. The person who wins the most rounds wins the game. Let $\frac{m}{n}$ be the probability that Emerson wins, where m and n are relatively prime positive integers. Find $m+n$.

30. **[25]** In a group of 6 people playing the card game Tractor, all 54 cards from 3 decks are dealt evenly to all the players at random. Each deck is dealt individually. Let the probability that no one has at least two of the same card be X. Find the largest integer n such that the nth root of X is rational.