LMT Spring Online: Division A

April 30th – May 7th, 2021

Contest Instructions

Contest Window

The Team Round consists of 30 short answer problems, including 10 theme problems. All answers are non-negative integers. The problems will be made available on the homepage of the LMT website on **Friday, April 30th, at 3:30 pm EDT**. Teams will have until **Friday, May 7th, at 11:59 pm EDT** to submit their answers using the link provided by email.

Contest Rules

With the exception of **standard four-function calculators**, computational aids including but not limited to scientific and graphing calculators, computer programs, and software such as Geogebra, Mathematica, and WolframAlpha, are **not** allowed. Communication of any form between students on different teams is similarly prohibited, and any team caught either giving or receiving an unfair advantage over other competitors will be disqualified. What constitutes cheating will be up to the final discretion of the competition organizers, who reserve the right to disqualify any team suspected of violating these rules.

Submitting Answers and Editing Team Information

During registration, your captain will be emailed a link to your team's homepage. This is where you will be able to update team information and answer submissions. We recommend that your captain distribute this link to the rest of the team so that the entire team has access. Once on your team's homepage, to enter or edit team answers, click the link next to "Submission Link:". Team name, team member names, and grades may be edited through the homepage as well. Remember, team member names must be the real names of the people on your team, and the team name must be appropriate.

Errata

If you believe there to be an error in one of the questions, email us at lmt@lhsmath.org with "Clarification" as the subject. Clarifications for problems will be updated on the LMT homepage, if necessary.

Scoring

The score of your team is the sum of the point values of the problems you answered correctly. **Note that the theme problems deviate from the general trend in point values.** Ties will not be broken. Results will be posted shortly after the competition, and the top teams will be recognized.

1. **[10]** Triangle LMT has $\overline{MA}$ as an altitude. Given that $MA = 16$, $MT = 20$, and $LT = 25$, find the length of the altitude from L to $\overline{MT}$.

2. **[10]** The function $f(x)$ has the property that $f(x) = -\frac{1}{f(x-1)}$. Given that $f(0) = -\frac{1}{21}$, find the value of $f(2021)$.

3. **[10]** Find the greatest possible sum of integers a and b such that $\frac{2021!}{20^a \cdot 21^b}$ is a positive integer.

4. **[10]** Five members of the Lexington Math Team are sitting around a table. Each flips a fair coin. Given that the probability that three consecutive members flip heads is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

5. **[15]** In rectangle $ABCD$, points E and F are on sides $\overline{BC}$ and $\overline{AD}$, respectively. Two congruent semicircles are drawn with centers E and F such that they both lie entirely on or inside the rectangle, the semicircle with center E passes through C, and the semicircle with center F passes through A. Given that $AB = 8$, $CE = 5$, and the semicircles are tangent, find the length BC.

6. **[15]** Given that the expected amount of 1s in a randomly selected 2021-digit number is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

7. **[15]** A geometric sequence consists of 11 terms. The arithmetic mean of the first 6 terms is 63, and the arithmetic mean of the last 6 terms is 2016. Find the 7th term in the sequence.

8. **[15]** Isosceles $\triangle ABC$ has interior point O such that $AO = \sqrt{52}$, $BO = 3$, and $CO = 5$. Given that $\angle ABC = 120^\circ$, find the length AB.

9. **[20]** Find the sum of all positive integers n such that $7 < n < 100$ and 1573_n has 6 factors when written in base 10.

10. **[20]** Pieck the Frog hops on Pascal's Triangle, where she starts at the number 1 at the top. In a hop, Pieck can hop to one of the two numbers directly below the number she is currently on with equal probability. Given that the expected value of the number she is on after 7 hops is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

11. **[20]** In $\triangle ABC$ with $\angle BAC = 60^\circ$ and circumcircle ω, the angle bisector of $\angle BAC$ intersects side $\overline{BC}$ at point D, and line AD is extended past D to a point A'. Let points E and F be the feet of the perpendiculars of A' onto lines AB and AC, respectively. Suppose that ω is tangent to line EF at a point P between E and F such that $\frac{EP}{FP} = \frac{1}{2}$. Given that $EF = 6$, the area of $\triangle ABC$ can be written as $\frac{m\sqrt{n}}{p}$, where m and p are relatively prime positive integers, and n is a positive integer not divisible by the square of any prime. Find $m+n+p$.

12. **[20]** There are 23 balls on a table, all of which are either red or blue, such that the probability that there are n red balls and $23-n$ blue balls on the table $(1 \le n \le 22)$ is proportional to n. (e.g. the probability that there are 2 red balls and 21 blue balls is twice the probability that there are 1 red ball and 22 blue balls.) Given that the probability that the red balls and blue balls can be arranged in a line such that there is a blue ball on each end, no two red balls are next to each other, and an equal number of blue balls can be placed between each pair of adjacent red balls is $\frac{a}{b}$, where a and b are relatively prime positive integers, find $a+b$.
Note: There can be any number of consecutive blue balls at the ends of the line.

13. **[25]** In a round-robin tournament, where any two players play each other exactly once, the fact holds that among every three students A, B, and C, one of the students beats the other two. Given that there are six players in the tournament and Aidan beats Zach but loses to Andrew, find how many ways there are for the tournament to play out.
Note: The order in which the matches take place does not matter.

14. **[25]** Alex, Bob, and Chris are driving cars down a road at distinct constant rates. All people are driving a positive integer number of miles per hour. All of their cars are 15 feet long. It takes Alex 1 second longer to completely pass Chris than it takes Bob to completely pass Chris. The passing time is defined as the time where their cars overlap. Find the smallest possible sum of their speeds, in miles per hour.

15. **[25]** Andy and Eddie play a game in which they continuously flip a fair coin. They stop flipping when either they flip tails, heads, and tails consecutively in that order, or they flip three tails in a row. Then, if there has been an odd number of flips, Andy wins, and otherwise Eddie wins. Given that the probability that Andy wins is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

16. **[25]** Find the number of ordered pairs (a, b) of positive integers less than or equal to 20 such that

$$\gcd(a,b) > 1 \quad \text{and} \quad \frac{1}{\gcd(a,b)} + \frac{a+b}{\operatorname{lcm}(a,b)} \geq 1.$$

17. **[30]** Given that the value of

$$\sum_{k=1}^{2021} \frac{1}{1^2+2^2+3^2+\cdots+k^2} + \sum_{k=1}^{1010} \frac{6}{2k^2-k} + \sum_{k=1011}^{2021} \frac{24}{2k+1}$$

can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

18. **[30]** Points X and Y are on a parabola of the form $y = \frac{x^2}{a^2}$ and A is the point $(x, y) = (0, a)$. Assume XY passes through A and hits the line $y = -a$ at a point B. Let ω be the circle passing through $(0, -a)$, A, and B. A point P is chosen on ω such that $PA = 8$. Given that X is between A and B, $AX = 2$, and $XB = 10$, find $PX \cdot PY$.

19. **[30]** Let S be the sum of all possible values of $a \cdot c$ such that

$$a^3 + 3ab^2 - 72ab + 432a = 4c^3$$

if a, b, and c are positive integers, $a+b > 11$, $a > b-13$, and $c \leq 1000$. Find the sum of all distinct prime factors of S.

20. **[30]** Let Ω be a circle with center O. Let ω_1 and ω_2 be circles with centers O_1 and O_2, respectively, internally tangent to Ω at points A and B, respectively, such that O_1 is on $\overline{OA}$, and O_2 is on $\overline{OB}$ and ω_1. There exists a point P on line AB such that P is on both ω_1 and ω_2. Let the external tangent of ω_1 and ω_2 on the same side of line AB as O hit ω_1 at X and ω_2 at Y, and let lines AX and BY intersect at N. Given that $O_1X = 81$ and $O_2Y = 18$, the value of $NX \cdot NA$ can be written as $a\sqrt{b} + c$, where a, b, and c are positive integers, and b is not divisible by the square of a prime. Find $a+b+c$.

Haikus

A Haiku is a Japanese poem of seventeen syllables, in three lines of five, seven, and five.

21. **[15]** In how many ways
Can you add three integers
Summing seventeen?

Order matters here.
For example, eight, three, six
Is not eight, six, three.

All nonnegative,
Do not need to be distinct.
What is your answer?

22. **[20]** Ada has been told
To write down five haikus plus
Two more every hour.

Such that she needs to
Write down five in the first hour
Seven, nine, so on.

Ada has so far
Forty haikus and writes down
Seven every hour.

At which hour after
She begins will she not have
Enough haikus done?

23. **[20]** A group of haikus
Some have one syllable less
Sixteen in total.

The group of haikus
Some have one syllable more
Eighteen in total.

What is the largest
Total count of syllables
That the group can't have?

(For instance, a group
Sixteen, seventeen, eighteen
Fifty-one total.)

24. **[25]** Using the four words
"Hi", "hey", "hello", and "haiku",
How many haikus

Can somebody make?
(Repetition is allowed,
Order does matter.)

Chandler the Octopus

25. **[15]** Chandler the Octopus is making a concoction to create the perfect ink. He adds 1.2 grams of melanin, 4.2 grams of enzymes, and 6.6 grams of polysaccharides. But Chandler accidentally added n grams of an extra ingredient to the concoction, Chemical X, to create glue. Given that Chemical X contains none of the three aforementioned ingredients, and the percentages of melanin, enzymes, and polysaccharides in the final concoction are all integers, find the sum of all possible positive integer values of n.

26. **[20]** Chandler the Octopus along with his friends Maisy the Bear and Jeff the Frog are solving LMT problems. It takes Maisy 3 minutes to solve a problem, Chandler 4 minutes to solve a problem and Jeff 5 minutes to solve a problem. They start at 12:00 pm, and Chandler has a dentist appointment from 12:10 pm to 12:30, after which he comes back and continues solving LMT problems. The time it will take for them to finish solving 50 LMT problems, in hours, is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$. **Note: they may collaborate on problems.**

27. **[25]** Chandler the Octopus is at a tentacle party!

    At this party, there is 1 creature with 2 tentacles, 2 creatures with 3 tentacles, 3 creatures with 4 tentacles, all the way up to 14 creatures with 15 tentacles. Each tentacle is distinguishable from all other tentacles. For some $2 \le m \ne n \le 15$, all creatures with m tentacles "meet" all creatures with n tentacles; "meeting" another creature consists of shaking exactly 1 tentacle with each other. Find the number of ways there are to pick distinct m and n between 2 and 15, inclusive, and then for all creatures with m tentacles to "meet" all creatures with n tentacles.

Card Games

28. **[15]** Addison and Emerson are playing a card game with three rounds. Addison has the cards 1, 3, and 5, and Emerson has the cards 2, 4, and 6. In advance of the game, both designate each one of their cards to be played for either round one, two, or three. Cards cannot be played for multiple rounds. In each round, both show each other their designated card for that round, and the person with the higher-numbered card wins the round. The person who wins the most rounds wins the game. Let $\frac{m}{n}$ be the probability that Emerson wins, where m and n are relatively prime positive integers. Find $m+n$.

29. **[20]** In a group of 6 people playing the card game Tractor, all 54 cards from 3 decks are dealt evenly to all the players at random. Each deck is dealt individually. Let the probability that no one has at least two of the same card be X. Find the largest integer n such that the nth root of X is rational.

30. **[25]** Ryan Murphy is playing poker. He is dealt a hand of 5 cards. Given that the probability that he has a straight hand (the ranks are all consecutive; e.g. $3,4,5,6,7$ or $9,10,J,Q,K$) or 3 of a kind (at least 3 cards of the same rank; e.g. $5,5,5$) is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.