LMT Spring Online

May 9th – May 15th, 2020

Contest Instructions

Contest Window

The competition consists of a single round, consisting of 30 short answer problems. All answers are non-negative integers. The problems will be made available on the homepage of the LMT website on **Saturday, May 9th, at 12:00 pm**. Teams will have until **Friday, May 15th, at 3:00 pm** to submit their answers using the link provided by email.

Contest Rules

With the exception of **standard four-function calculators**, computational aids including but not limited to scientific and graphing calculators, computer programs, and software such as Geogebra, Mathematica, and WolframAlpha, are **not** allowed. Communication of any form between students on different teams is similarly prohibited, and any team caught either giving or receiving an unfair advantage over other competitors will be disqualified. What constitutes cheating will be up to the final discretion of the competition organizers, who reserve the right to disqualify any team suspected of violating these rules.

Submitting Answers and Editing Team Information

During registration, your captain will be emailed a link to your team's homepage. This is where you will be able to update team information and answer submissions. We recommend that your captain distribute this link to the rest of the team so that the entire team has access. Once on your team's homepage, to enter or edit team answers, click the link next to "Submission Link:". Team name, team member names, and grades may be edited through the homepage as well. Remember, team member names must be the real names of the people on your team, and the team name must be appropriate.

Errata

If you believe there to be an error in one of the questions, email us at lmt@lhsmath.org with "Clarification" as the subject. Clarifications for problems will be updated on the LMT homepage, if necessary.

Scoring

The score of your team is the number of questions you answer correctly. We will break ties by weighing the problems based on how many teams solve them. Results will be posted shortly after the competition, where the top teams will be recognized.

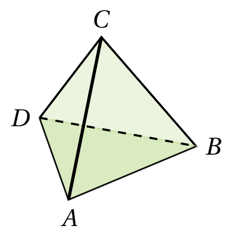
1. Compute the smallest nonnegative integer that can be written as the sum of 2020 distinct integers.

   Proposed by Janabel Xia

   Solution. $\boxed{0}$

   There are an infinite number of possibilities for the set of 2020 distinct integers such that their sum is 0. For example, take the subset $S = -1010, -1009, \ldots, -2, -1, 1, 2, \ldots, 1009, 1010$. □

2. In tetrahedron $ABCD$, as shown below, compute the number of ways to start at A, walk along some path of edges, and arrive back at A without walking over the same edge twice.



   Proposed by Richard Chen

   Solution. $\boxed{12}$

   Since we cannot walk over the same edge twice, we cannot only visit one vertex on triangle BCD, as that would require traveling to and from triangle BCD along the same edge. Thus we can visit either 2 or 3 vertices on triangle BCD. If we visit 2, there are 3 choices for the first vertex and 2 choices for the second, giving $3 \cdot 2 = 6$ ways to walk. If we visit 3, there are 3 choices for the first vertex, 2 choices for the second, and 1 choice for the third, giving us $3 \cdot 2 \cdot 1 = 6$ ways to walk. In total, there are $6 + 6 = \boxed{12}$ ways. □

3. Let LMT represent a 3-digit positive integer where L and M are nonzero digits. Suppose that the 2-digit number MT divides LMT. Compute the difference between the maximum and minimum possible values of LMT.

   Proposed by Janabel Xia

   Solution. $\boxed{880}$

   Note that if $MT \mid LMT$, then $MT \mid 100L$. For the minimizing LMT, we first minimize L as $L = 1$, which gives us $MT \mid 100$ for a minimum value of $MT = 10$. For maximizing LMT, letting $L = 9$ gives us $MT \mid 900$ for a maximum value of $MT = 90$. Then our desired difference is $990 - 110 = \boxed{880}$. □

4. Suppose there are n ordered pairs of positive integers (a_i, b_i) such that $a_i + b_i = 2020$ and $a_i b_i$ is a multiple of 2020, where $1 \le i \le n$. Compute the sum

$$\sum_{i=1}^{n} a_i + b_i.$$

   Proposed by Alex Li

   Solution. $\boxed{2020}$

   The prime factorization of 2020 is $2^2 \cdot 5 \cdot 101$. Since $2020 \mid a_i b_i$, at least one of either a_i or b_i must be a multiple of 2, 5, and 101. Moreover, since $a_i + b_i = 2020$, both a_i and b_i must be multiples of 2, 5, and 101. It follows that $1010 \mid a_i$ and $1010 \mid b_i$, and since $a_i + b_i = 2020$, the only ordered pair that satisfies these requirements is $(a_1, b_1) = (1010, 1010)$, which gives $a_1 + b_1 = \boxed{2020}$. □

5. For a positive integer n, let $\mathscr{D}(n)$ be the value obtained by, starting from the left, alternating between adding and subtracting the digits of n. For example, $\mathscr{D}(321) = 3-2+1 = 2$, while $\mathscr{D}(40) = 4-0 = 4$. Compute the value of the sum

$$\sum_{n=1}^{100} \mathscr{D}(n) = \mathscr{D}(1) + \mathscr{D}(2) + \cdots + \mathscr{D}(100).$$

Proposed by Ezra Erives

Solution. $\boxed{91}$

We start with one-digit numbers. The numbers one through nine contribute $1+2+\cdots+9 = 45$ to our total. For two-digit numbers, we notice that each digit is added ten times as a tens digit, and subtracted nine times as a ones digit, and so our total increases by $1+2+\cdots+9 = 45$. Finally, 100 adds one to this total, and so our answer is $45+45+1 = \boxed{91}$. □

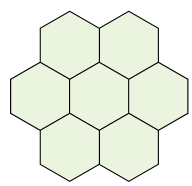
6. Let $\triangle ABC$ be a triangle such that $AB = 6, BC = 8$, and $AC = 10$. Let M be the midpoint of BC. Circle ω passes through A and is tangent to BC at M. Suppose ω intersects segments AB and AC again at points X and Y, respectively. If the area of AXY can be expressed as $\frac{p}{q}$ where p, q are relatively prime integers, compute $p+q$.

Proposed by Janabel Xia

Solution. $\boxed{61}$

Using Power of a Point, we have $BX \cdot BA = BM^2 \Rightarrow BX = \frac{4^2}{6} = \frac{8}{3}$ and $CY \cdot CA = CM^2 \Rightarrow CY = \frac{4^2}{10} = \frac{8}{5}$. Then we can calculate $[AXY] = [ABC] \cdot \frac{AX}{AB} \cdot AY\,AC = 24 \cdot \frac{10}{18} \cdot \frac{42}{50} = \frac{56}{5}$, giving us $56+5 = \boxed{61}$ as our final answer. □

7. The hexagonal pattern constructed below has two smaller hexagons per side and has a total of 30 edges. A similar figure is constructed with 20 smaller hexagons per side. Compute the number of edges in this larger figure.



Proposed by Ezra Erives

Solution. $\boxed{3540}$

Let H_{20}, B_{20}, and E_{20} be the total number of hexagons in the figure, the number of boundary edges in the figure, and the total number of edges in the figure respectively. Then $6H_{20} = 2E_{20} - B_{20}$, and thus $E_{20} = \frac{6H_{20}+B_{20}}{2}$. We compute that $H_{20} = 20+21+\cdots+39+\cdots+21+20 = 1141$, and $B_{20} = 6 \cdot 19 \cdot 2 + 6 = 234$, and so

$$E_{20} = \frac{6 \cdot 1141 + 234}{2} = \boxed{3540}.$$

□

8. Let a, b be real numbers satisfying $a^2 + b^2 = 3ab = 75$ and $a > b$. Compute $a^3 - b^3$.

Proposed by Richard Chen

Solution. $\boxed{500}$

From the given expressions regarding a and b, we see that $a^2 + b^2 = 75$ and $ab = 25$. Thus $(a-b)^2 = a^2 + b^2 - 2ab = 75 - 2 \cdot 25 = 25$, so $a - b = 5$ since $a > b$. By difference of cubes factorization, $a^3 - b^3 = (a-b)(a^2+ab+b^2) = 5(75+25) = \boxed{500}$. □

9. A function $f(x)$ is such that for any integer x, $f(x) + xf(2-x) = 6$. Compute $-2019f(2020)$.

Proposed by Ezra Erives

Solution. $\boxed{6}$

Notice first off that the only sensible values of x to plug in are 2020 and −2018. Plugging them in yields

$$f(2020) + 2020f(-2018) = 6$$
$$f(-2018) - 2018f(2020 = 6.$$

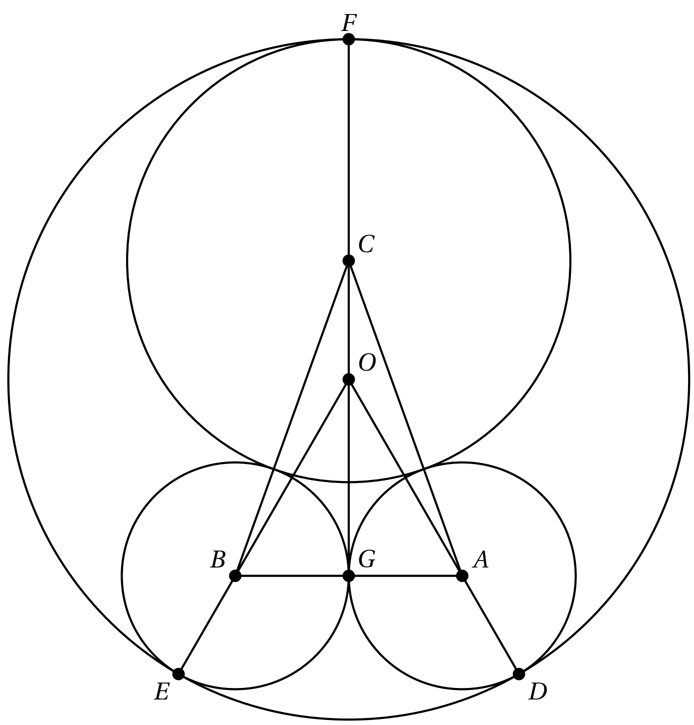
We have two equations in two unknowns. Subtracting the two equations gives $2019(f(2020) + f(-2018)) = 0$, and thus $f(-2018) = -f(2020)$. Plugging back into the first equation gives $-2019f(2020) = 6$, and so the desired answer is $\boxed{6}$.

□

10. Three mutually externally tangent circles are internally tangent to a circle with radius 1. If two of the inner circles have radius $\frac{1}{3}$, the largest possible radius of the third inner circle can be expressed in the form $\frac{a+b\sqrt{c}}{d}$ where c is squarefree and $\gcd(a,b,d) = 1$. Find $a+b+c+d$.

Proposed by Alex Li

Solution. $\boxed{23}$



Let O be the center of the circle with radius 1, A and B the centers of the circles with radii $\frac{1}{3}$, and C the center of the circle with unknown radius r. We see that $OF = OE = OD = 1$, and since $CF = r$ and $BE = AD = \frac{1}{3}$, we have $OB = OA = \frac{2}{3}$ and $OC = 1 - r$. Focusing on triangle ABC, observe that triangle ABO is equilateral with $AB = BO = AO = \frac{2}{3}$. This gives $GO = \frac{\sqrt{3}}{3}$, so $GC = \frac{\sqrt{3}}{3} + 1 - r$. Since $AC = \frac{1}{3} + r$, from the Pythagorean theorem on triangle AGC,

$$\left(\frac{1}{3}\right)^2 + \left(\frac{\sqrt{3}}{3} + 1 - r\right)^2 = \left(\frac{1}{3} + r\right)^2.$$

This simplifies to $r = \frac{2+\sqrt{3}}{4+\sqrt{3}} = \frac{5+2\sqrt{3}}{13}$, which gives $5+2+3+13 = \boxed{23}$ as the answer. □

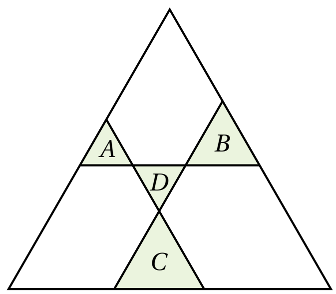
11. Let set $\mathscr{S}$ contain all positive integers less than or equal to 2020 that can be written in the form $n(n+1)$ for some positive integer n. Compute the number of ordered pairs (a,b) such that $a,b \in \mathscr{S}$ and $a-b$ is a power of two.

Proposed by Alex Li

Solution. $\boxed{5}$

Suppose $a = m(m+1) = m^2+m$ and $b = n(n+1) = n^2+n$ satisfy the desired condition; that is, $a-b = m^2-n^2+m-n = 2^k$ for some nonnegative integer k. Rearranging and factoring a difference of squares gives $2^k = (m+n)(m-n)+(m-n) = (m+n+1)(m-n)$. It follows that both $m+n+1$ and $m-n$ must be powers of two. However, since $m+n+1 \equiv m-n+1 \not\equiv m-n \pmod 2$, we must either have $m+n+1 = 1$ or $m-n = 1$. Clearly the former cannot be true since a and b must be positive; thus $m = n+1$. Substituting for n gives $2m = 2^k \Rightarrow m = 2^{k-1}$, which means m must be a power of two. We see that if $m = 1$, then $n = 0$ and $b = 0$, which is not in set $\mathscr{S}$, and if $m = 64$, then $a = 4160 > 2020$ which is also not in the set. These bounds give us possible values of m as $2,4,8,16,32$ with corresponding values of a as $6,20,72,272,1056$. The corresponding values for b are $2,12,56,240,992$, which are all in set $\mathscr{S}$. Thus there are $\boxed{5}$ valid ordered pairs. □

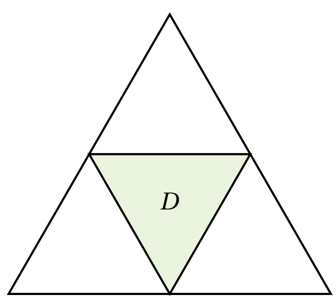
12. In the figure above, the large triangle and all four shaded triangles are equilateral. If the areas of triangles A, B, and C are 1, 2, and 3, respectively, compute the smallest possible integer ratio between the area of the entire triangle to the area of triangle D.



Proposed by Alex Li

Solution. $\boxed{5}$

Solution 1: Consider a situation where A, B, and C are much smaller than D, such as shown above. It is clear that the



desired ratio is very close to 4 in this case. However, since the sizes of A, B, and C are still finite, the actual ratio is slightly larger than 4, no matter how small A, B, and C become. Thus the smallest integer ratio is $\boxed{5}$.

Solution 2: Let the side lengths of triangles A,B,C,D be a,b,c,d, respectively. One can deduce the side length of the large equilateral triangle is $a+b+c+2d$ by observing the side lengths of the equilateral triangles containing A and B, B and C, and C and D are $a+b+d$, $b+c+d$, and $c+a+d$, respectively. Thus the ratio of the side length of the large equilateral triangle to that of triangle D is $\frac{a+b+c+2d}{d} = 2+\frac{a+b+c}{d}$. The ratio of the areas is simply the square of the ratio of the side lengths, or $\left(2+\frac{a+b+c}{d}\right)^2$. One can observe that as d becomes much larger than $a+b+c$, the desired ratio approaches $2^2=4$. However, since this value can never be obtained since $a+b+c>0$, the smallest possible integer ratio is thus $\boxed{5}$. □

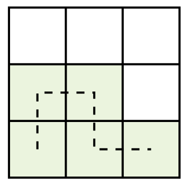
13. In the game of *Flow*, a path is drawn through a 3×3 grid of squares obeying the following rules:

   i A path is continuous with no breaks (it can be drawn without lifting a pencil).

   ii A path that spans multiple squares can only be drawn between colored squares that share a side.

   iii A path cannot go through a square more than once.

   Compute the number of ways to color a positive number of squares on the grid such that a valid path can be drawn. An example of one such coloring and a valid path is shown below.



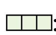
Proposed by Alex Li

Solution. $\boxed{171}$

We organize our cases by the size and shape of the coloring:

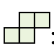
: There are clearly 9 such colorings.

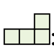
: These colorings can be oriented horizontally or vertically. Each case gives 6 colorings, which gives a total of 12 colorings.

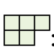
: Each such coloring corresponds to either a row or column, giving 6 colorings.

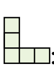
: These colorings occupy a 2×2 space with one of the four corners missing. Since there are 4 ways to choose a 2×2 space on the 3×3 board, there are $4\cdot 4=16$ colorings.

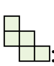
: There are 4 ways to choose a 2×2 space on the 3×3 board, so there are 4 colorings.

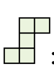
: The coloring must be along an outside row/column. For each outside row/column, there are two ways to orient the coloring. This gives $4\cdot 2=8$ colorings.

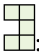
: The long edge can either be along an outside row/column or an inside row/column. If it is along an outside row/column, there are 2 ways to orient the coloring, and if its along an inside row/column, there are 4 ways to orient the coloring. This gives $4\cdot 2+2\cdot 4=16$ colorings.

: Each coloring from the previous case can be modified by adding a square to give a coloring from this case. There are also 16 colorings.

: The corner piece must coincide with a corner of the grid. The 4 corners gives 4 colorings.

: Each coloring from the previous case can be modified by moving the corner piece to give a coloring from this case. There are also 4 colorings.

: The coloring must be oriented along an inside row/column. Along each of them, there are two distinct orientations of the coloring. This gives $2\cdot 2=4$ colorings.

: The coloring must be oriented along a row/column. If it is oriented along an interior row/column there are two possible orientations. This gives $4+4=8$ colorings.

At this point, it is easier to count the uncolored squares rather than the colored squares. Note that a different location on the board could affect whether or not a valid path could be formed.

If the path has length 9, every square is colored. There is only 1 coloring.

If the path has length 8, the uncolored square must be the center or a corner square. This gives 5 colorings.

If the path has length 7, the two uncolored squares can be next to each other. This gives 12 colorings. If they are not, they can be at two corners, diagonally adjacent, or a knights move away from each other. The first of these gives 6 colorings; 4 for adjacent corners and 2 for opposite corners. If they are diagonally adjacent, one must be at the center, and the other can be at one of 4 corners, giving 4 colorings. If they are a knights move away, one uncolored square must be a corner square, and the other can be one of two edge squares. Since there are 4 corners, this gives 8 colorings, giving $12+6+4+8=30$ total colorings.

If the path has length 6, the three uncolored squares can be next to each other. This gives 4 colorings. The three uncolored squares can also form an L-shape, given one of the two edge squares occupies the center square. This gives 8 cases, two for each corner. If two are next to each other and one is not, there are two such arrangements, each of which has 8 colorings: the two adjacent uncolored squares must lie on a side, each of which has two orientations. This gives $4+8+8+8=28$ total colorings.

Summing all these cases yields a total of $\boxed{171}$.

□

14. Let $\triangle ABC$ be a triangle such that $AB=40$ and $AC=30$. Points X and Y are on the segment AB and BC, respectively such that $AX:BX=3:2$ and $BY:CY=1:4$. Given that $XY=12$, the area of $\triangle ABC$ can be written as $a\sqrt{b}$ where a and b are positive integers and b is squarefree. Compute $a+b$.

Proposed by Sooyoung Choi

Solution. $\boxed{480}$

Solution 1: Our motivation in any Geometry problems for LMT is that we never bash.

Notice that the length of XY is 12 which is $\frac{2}{5}$ of AC. Let Z be the point on BC such that $BY=YZ$. Notice that $XZ=12$ since XZ is parallel to AC. Then $\triangle XYZ$ is isosceles. Then, we drop the altitude from X to BC. Let $BY=x$. By Pythagorean theorem, we have $16^2-(\frac{3}{2}x)^2=12^2-(\frac{1}{2}x)^2$. Solving for x gives $x=2\sqrt{14}$. Then, the area of $\triangle XYZ$ is $2\sqrt{455}$ and using the ratio, the area of $\triangle ABC$ equals to $25\sqrt{455}$ so our answer is $455+25=\boxed{480}$.

Solution 2: Although our motivation is no bash, we still present a trashy bashy solution.

We use the Law of Cosines. We have $BX=\frac{2}{5}AB=16$, and let $BY=n$, $BC=5n$. Then from the Law of Cosines on $\triangle BXY$ and $\triangle BAC$ respectively, we have $XY^2=16^2+n^2-2(16)(n)(\cos B)=12^2$ and $AC^2=40^2+(5n)^2-2(40)(5n)(\cos B)=30^2$. Multiplying the first equation by 5^2 and the second by 2 and subtracting lets us solve for $n=2\sqrt{14}$. Substituting this into either equation lets us solve for $\cos B=\frac{3\sqrt{14}}{16}$. Using the relation $\sin^2(x)+\cos^2(x)=1$ for all x, we get $\sin B=\frac{\sqrt{130}}{16}$. Finally, $[ABC]=\frac{1}{2}(BA)(BC)(\sin B)=25\sqrt{455}$, so $a+b=25+455=\boxed{480}$ is our desired answer.

□

15. Let $\phi(k)$ denote the number of positive integers less than or equal to k that are relatively prime to k. For example, $\phi(2)=1$ and $\phi(10)=4$. Compute the number of positive integers $n\le 2020$ such that $\phi(n^2)=2\phi(n)^2$.

Proposed by Janabel Xia

Solution. $\boxed{10}$

Let n be a number satisfying this equation, and let its the prime factorization be $p_1^{k_1} \cdot p_2^{k_2} \cdot \dots \cdot p_i^{k_i}$.

Claim: $\phi(n) = \prod_{j=1}^{i}(p_j - 1)\left(p_j^{k_j-1}\right)$

Proof: We can look at each p_j individually, since a number is relatively prime if it shares no prime divisors. The probability any chosen number $a < n$ is not divisible by p_j for each j is $P(p_j \nmid a) = \frac{p_i - 1}{p_i}$, given that $p_i \mid n$, and divisibility by each p_i of a random chosen number is independent. Thus, we have

$$\phi(n) = nP(p_1, p_2, \dots, p_i \nmid a) = n\prod_{j=1}^{i} P(p_j \nmid a) = n\prod_{j=1}^{i}\frac{p_j - 1}{p_j} = \prod_{j=1}^{i}(p_j - 1)\left(p_j^{k_j-1}\right).$$

Now we have

$$\phi(n^2) = \prod_{j=1}^{i}(p_j - 1)\left(p_j^{2k_j-1}\right)$$

and

$$2\phi(n)^2 = 2\Big(\prod_{j=1}^{i}(p_j - 1)(p_j^{k_j-1}\Big)^2 = 2\prod_{j=1}^{i}(p_j - 1)^2\left(p_j^{2k_j-2}\right)$$

are equal, and canceling out (lots!) of terms gives us

$$\frac{1}{2} = \prod_{j=1}^{i}\frac{p_j - 1}{p_j}.$$

Note that p_i cannot be canceled from the denominator of the RHS, as it is the largest prime divisor, unless $p_i = 2$ already. Thus, $n \le 2020$ is be a power of 2 greater than 1, giving us $\boxed{10}$ total possibilities. □

16. For non-negative integer n, the function f is given by

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } n \text{ is even} \\ x - 1 & \text{if } n \text{ is odd.} \end{cases}$$

Furthermore, let $h(n)$ be the smallest k for which $f^k(n) = 0$. Compute

$$\sum_{n=1}^{1024} h(n).$$

Proposed by Ezra Erives

Solution. $\boxed{13325}$

This problem becomes easier by considering the numbers in binary. Dividing by two removes a zero from the end of a binary string, if that string initially ended with a zero (the string was an even number). Subtracting one removes a one from the end of a binary string (the string was an odd number). These two observations together are enough to imply that $h(n)$ is equal to the number of ones in the binary representation of n plus the number of digits in the binary representation minus one. In this sense, $h(n)$ can be thought of as $h(n) = t(n) + d(n) - 1$, where $t(n)$ is the number of ones in the binary representation of n and $d(n)$ is the number of digits in the binary representation of n. We perform the summations separately. We have

$$\sum_{n=1}^{1024} t(n) = 1 + \sum_{k=1}^{10} k\binom{10}{k} = 5121$$

and

$$\sum_{n=1}^{1024} d(n) = 11 + \sum_{k=1}^{10} k2^{k-1} = 9228.$$

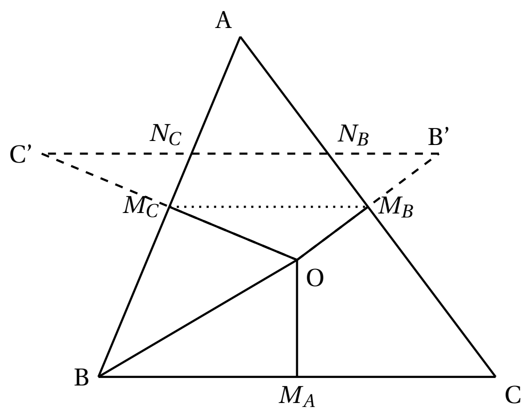
It follows that our answer is $5121 + 9228 - 1024 = \boxed{13325}$. □

17. Let ABC be a triangle such that $AB = 26$, $AC = 30$, and $BC = 28$. Let C' and B' be the reflections of the circumcenter O over AB and AC, respectively. The length of the portion of line segment $B'C'$ inside triangle ABC can be written as $\frac{p}{q}$, where p, q are relatively prime positive integers. Compute $p+q$.

    Proposed by Richard Chen

    Solution. $\boxed{85}$

    We take advantage of all of the similar triangles involved in the picture.



Notice that $\triangle OM_CM_B \sim \triangle OC'B'$. Further note that $\triangle AN_CN_B \sim \triangle AM_CM_B \sim \triangle ABC$. Comparing the heights of these these triangles will do.

We have that the length of the height from A to BC is 24(notice the $26-24-10$ and $30-24-18$ right triangles formed!) and that the length of the height from A to M_CM_B is $24 \div 2 = 12$. We now want to find the height from A to N_CN_B, which is $12-$ (height from $C'B'$ to M_CM_B). Notice that the height from $C'B'$ to M_CM_B is equal to the height from O to M_CM_B, by similar triangles $\triangle OM_CM_B$ and $\triangle OC'B'$. Thus, the height from A to M_CM_B is equal in length to $12-$ (height from O to M_CM_B), which is equal in length to OM_A.

To find the length of OM_A, we look at triangle OM_AB; this is a right triangle with legs $BM_A = 14$ and OM_A and hypotenuse equal to the circumradius of the triangle. Using the circumradius formula, $R = \frac{abc}{4[\text{area}]} \implies R = \frac{65}{4}$, we then finish with Pythagorean theorem to determine that OM_A is $\frac{33}{4}$, so the height from A to M_CM_B is $\frac{33}{4}$.

We now have similar triangles; the similar triangle we are looking for, AN_CN_B, can be compared to ABC using height; we get that the scale factor here is $\frac{33}{4} \div 24 = \frac{11}{32}$, so $N_CN_B = \frac{11}{32} \cdot BC = \frac{11}{32} \cdot 28 = \frac{77}{8}$, for a final answer of $77+8 = \boxed{85}$.

□

18. Compute the maximum integer value of k such that 2^k divides $3^{2n+3} + 40n - 27$ for any positive integer n.

    Proposed by Sooyoung Choi

    Solution. $\boxed{6}$

    Our strategy is to first try to factor the expression. Seeing the 3 in the exponent of the power of 3 and the constant 27 term, we group them together to get the following factorization:

$$\begin{aligned} &3^{2n+3} + 40n - 27 \\ &= 3^3(3^{2n}-1) + 40n \\ &= 3^3(9^n - 1) + 40n \\ &= 3^3(9-1)(9^{n-1} + 9^{n-2} + \cdots + 1) + 40n \\ &= 8(3^3(9^{n-1} + 9^{n-2} + \cdots + 1) + 5n), \end{aligned}$$

which gives us a factor of $8 = 2^3$ already. We also have

$$3^3(9^{n-1}+9^{n-2}+\cdots+1)+5n \equiv 27n+5n \equiv 32n \equiv 0 \pmod 8,$$

which gives us another factor of 2^3. Finally, plugging in $n=2$ gives us $3^3(9^{n-1}+9^{n-2}+\cdots+1)+5n = 280$, which has exactly 3 factors of 2. Therefore, our answer is $k = 3+3 = \boxed{6}$ □

19. Let ABC be a triangle such that such that $AB = 14, BC = 13$, and $AC = 15$. Let X be a point inside triangle ABC. Compute the minimum possible value of $(\sqrt{2}AX+BX+CX)^2$.

Proposed by Sooyoung Choi

Solution. $\boxed{757}$

Seeing the $\sqrt{2}$ in the expression we wish to minimize, we can try interpreting this as the leg of a $45-45-90$ triangle. To easily create such a triangle, we rotate $\triangle ABC$ $90°$ clockwise about A to triangle $AB'C'$, further letting the image of point X be X'. Then we have $(\sqrt{2}AX+BX+CX)^2 = (X'X+B'X'+XC)^2$, which is minimized at $B'C^2$ when B', X', X, C are collinear. To calculate $B'C^2$ easily, let the foot of the $C-$altitude be D. Then both $B'A \perp AB$ and $CD \perp AB$, so Pythagorean gives us $B'C^2 = (B'A+DC)^2 + AD^2 = 26^2+9^2 = \boxed{757}$. □

20. Let $c_1 < c_2 < c_3$ be the three smallest positive integer values of c such that the distance between the parabola $y = x^2+2020$ and the line $y = cx$ is a rational multiple of $\sqrt{2}$. Compute $c_1+c_2+c_3$.

Proposed by Ezra Erives and Nathan Ramesh

Solution. $\boxed{49}$

Let the line $y = cx+d$ be tangent to the parabola $y = x^2+2020$. It follows that $x^2+2020 = cx+d$ must have one solution, which is the case when the discriminant $c^2-4(2020-d) = 0$. This gives $d = 2020-\frac{c^2}{4}$, which is also the vertical distance between the lines $y = cx$ and $y = cx+d$. Since the line has slope c, the horizontal distance between the two lines is $\frac{d}{c}$. Similar triangles gives the shortest distance between the two lines as $\frac{\frac{d^2}{c}}{d\sqrt{1+\frac{1}{c^2}}} = \frac{d}{\sqrt{1+c^2}}$. Since this expression is a rational multiple of $\sqrt{2}$, it must be that $\frac{d}{\sqrt{1+c^2}} = r\sqrt{2}$ for some positive rational r. Rearranging and squaring gives $2c^2+2 = \left(\frac{d}{r}\right)^2$, and since c is an integer, the quotient $\frac{d}{r} = n$ for some positive integer n. (Note the dependence of d upon c can be neglected, since the rationality of d is the only important detail for solving the problem. The equation will be made true by the choice for r.) Equivalently, $2c^2+2$ must be a perfect square, and checking positive integers gives $1, 7$, and 41 as the three smallest values for which this is true. It follows that the answer is $1+7+41 = \boxed{49}$. □

21. Let $\{a_n\}$ be the sequence such that $a_0 = 2019$ and

$$a_n = -\frac{2020}{n}\sum_{k=0}^{n-1} a_k.$$

Compute the last three digits of $\sum_{n=1}^{2020} 2020^n a_n n$.

Proposed by Sooyoung Choi

Solution. $\boxed{400}$

Notice that we have $na_n = -2020\sum_{k=0}^{n-1} a_k$ and $(n-1)a_{n-1} = -2020\sum_{k=0}^{n-2} a_k$. Subtracting the two equations gives us

$$a_n = -\frac{(2020-(n-1))}{n}a_{n-1}.$$

Now using the above equation repeatedly and the fact that $a_0 = 2019$, we get $a_n = (-1)^n\binom{2020}{n}$. Then, $n\cdot 2020^n \cdot a_n = 2019\cdot(-2020)^n\cdot\binom{2020}{n}\cdot n = 2019\cdot(-2020)^n\cdot\frac{2020}{n}\cdot n\cdot\binom{2019}{n-1} = 2019\cdot 2020\cdot(-2020)\cdot(-2020)^{n-1}\binom{2019}{n-1}$.

Hence, $\sum_{n=1}^{2020} n\cdot 2020^n\cdot a_n = 2019\cdot 2020\cdot(-2020)\sum_{n=1} 2020(-2020)^{n-1}\binom{2019}{n-1} = 2019\cdot 2020\cdot(-2020)\sum_{n=0}^{2019}(-2020)^n\binom{2019}{n}$ $= 2019\cdot 2020\cdot(-2020)\cdot(1-2020)^{2019} = 2019^{2020}\cdot 2020^2$, finally giving us a remainder of $\boxed{400}$ when divided by 1000. □

22. The numbers one through eight are written, in that order, on a chalkboard. A mysterious higher power in possession of both an eraser and a piece of chalk chooses three distinct numbers x, y, and z on the board, and does the following. First, x is erased and replaced with y, after which y is erased and replaced with z, and finally z is erased and replaced with x. The higher power repeats this process some finite number of times. For example, if $(x, y, z) = (2,4,5)$ is chosen, followed by $(x, y, z) = (1,4,3)$, the board would change in the following manner:

$$12345678 \to 14352678 \to 43152678$$

Compute the number of possible final orderings of the eight numbers.

Proposed by Ezra Erives

Solution. $\boxed{20160}$

We define a *transposition* to be the action of swapping two elements of a permutation. For example, the trasposition corresponding to swapping the second and third elements, and referred to as (23), is shown below.

$$12345678 \to 13245678$$

Any permutation can be expressed as a product of some number of permutations, and can be done so in more than one way. It is not hard to show however that the parity of the number of transpositions needed to express a given permutation is invariant.

We claim that the set of possible final permutations is precisely the set of permutations that require an even number of permutations and which we will henceforth refer to as *even* permutations.

The mysterious higher power is only able to act on the permutation by cycling triplets of indices. The important observation is that cycling the elements at positions x, y, z (that is, where x goes to y goes to z goes to x) can be written as the product of transpositions (xy) and (yz). In other words, we can obtain the desired cycling of x, y, z by first swapping x and y, and then swapping y and z. It follows that any possible final permutation must be even.

We now show that every even permutation is achievable. Note that, for distinct indices a, b, c, d, the product $(ab)(cd)$ of transpositions (first swap c and d, and then swap a and b) can be achieved by the higher power by first cycling $b \to c \to d$ and then cycling $a \to b \to c$. Similar "factorizations" exist when a, b, c, d are not all distinct. Since any even permutation can be written as the product of an even number of transpositions, and each pair of transpositions can be expressed in terms of the mysterious higher powers "3-cycles", we conclude that all even permutations are achievable.

Consider some permutation. If this permutation is even, it can be achieved. If this permutation is not even (odd), the permutation obtained by swapping the last two elements is an even permutation. Thus, the entire set of permutations can be broken into $\frac{8!}{2} = \boxed{20160}$ pairs of permutations, with each pair containing exactly one even permutation. □

23. Let $\triangle ABC$ be a triangle such that $AB = AC = 40$ and $BC = 79$. Let X and Y be the points on segments AB and AC such that $AX = 5$, $AY = 25$. Given that P is the intersection of lines XY and BC, compute $PX\cdot PY - PB\cdot PC$.

Proposed by Sooyoung Choi

Solution. $\boxed{525}$

If you bashed this problem, we are disappointed. As for the solution, we are motivated to try to construct similar triangles from the expression given. Let point Q be on ray CB such that $\angle XQP = \angle CYP \Rightarrow \triangle PCY \sim \triangle PXQ$. This gives us $PX \cdot PY = PQ \cdot PC$, so $PX \cdot PY - PB \cdot PC = PQ \cdot PC - PB \cdot PC = BQ \cdot PC$. However, we also have $\angle XBQ = \angle PCY$ since $\triangle ABC$ is isosceles, so $\triangle PCY \sim \triangle XBQ$. Finally, similar triangle ratios gives us $\frac{PC}{CY} = \frac{XB}{BQ} \Rightarrow BQ \cdot PC = CY \cdot XB = 15 \cdot 35 = \boxed{525}$.

Remark from the author: As can be seen from our solution, the desired expression $PX \cdot PY - PB \cdot PC$ is constant and the length $BC = 79$ is irrelevant. Also, this problem is exclusively written for people who have May 25th as their birthday. □

24. Let a, b, and c be real angles such that

$$3\sin a + 4\sin b + 5\sin c = 0$$
$$3\cos a + 4\cos b + 5\cos c = 0.$$

The maximum value of the expression $\frac{\sin b \sin c}{\sin^2 a}$ can be expressed as $\frac{p}{q}$ for relatively prime p, q. Compute $p + q$.

Proposed by Alex Li

Solution. $\boxed{89}$

Multiplying the first equation by i and adding it to the second gives $3\operatorname{cis} a + 4\operatorname{cis} b + 5\operatorname{cis} c = 0$. Geometrically, the sum indicates that the complex numbers $3\operatorname{cis} a$, $4\operatorname{cis} b$, and $5\operatorname{cis} c$ must form the sides of a $3-4-5$ right triangle, with angles $b = a + 90$ and $c = 180 + \tan^{-1}(\frac{4}{3}) + a$. Since $\sin b = \sin(a+90) = \cos a$ and $\sin c = \sin\left(180 + \tan^{-1}(\frac{4}{3}) + a\right) = -\sin\left(\tan^{-1}(\frac{4}{3}) + a\right) = -\left(\sin\left(\tan^{-1}(\frac{4}{3})\right)\cos(a) - \cos\left(\tan^{-1}(\frac{4}{3})\right)\sin(a)\right) = -\left(\frac{4}{5}\cos a + \frac{3}{5}\sin a\right)$, the desired expression can be written as $-\left(\frac{4}{5}\cot^2 a + \frac{3}{5}\cot a\right)$. This expression is a quadratic in $\cot a$ and can be expressed equivalently as $-\frac{4}{5}\left(\cot a + \frac{3}{8}\right)^2 + \frac{9}{80}$. It follows that the maximum value is $\frac{9}{80}$, achieved when $\cot a = -\frac{3}{8}$, giving the desired sum as $\boxed{89}$. □

25. Let $\triangle ABC$ be a triangle such that $AB = 5$, $AC = 8$, and $\angle BAC = 60°$. Let Γ denote the circumcircle of ABC, and let I and O denote the incenter and circumcenter of $\triangle ABC$, respectively. Let P be the intersection of ray IO with Γ, and let X be the intersection of ray BI with Γ. If the area of quadrilateral $XICP$ can be expressed as $\frac{a\sqrt{b}+c\sqrt{d}}{e}$, where a and d are squarefree positive integers and $\gcd(a,c,e) = 1$, compute $a+b+c+d+e$.

Proposed by Janabel Xia

Solution. $\boxed{41}$

The key to solving this problem is to notice that XIC is an equilateral triangle symmetric about IO. Our first useful claim is that B, I, O, C are cyclic.

Proof: We have $\angle BIC = 180° - (\angle IBC + \angle ICB) = 180° - \frac{1}{2}(\angle B + \angle C) = 180° - \frac{1}{2}(180° - \angle A) = 120°$ and $\angle BOC = 2\angle A = 120°$, so $\angle BIC = \angle BOC \Rightarrow B, I, O, C$ concyclic.

Now we can angle chase as follows: $\angle OIC = \angle OBC = 30°$ (since $OB = OC = R$ circumradius) and $\angle OIX = 180° - \angle OIB = \angle OCB = 30°$. Therefore, $\angle XIC = 60°$ and XIC is symmetric about the diameter line $IO \Rightarrow XI = XC \Rightarrow XIC$ is equilateral and $XC \perp IQ$. Then, we can find $[XICQ] = \frac{1}{2} \cdot XC \cdot IQ$.

First, we have $BC^2 = AB^2 + AC^2 - 2 \cdot AB \cdot AC \cdot \cos(A) = 49 \Rightarrow BC = 7$ from Law of Cosines. To find $XC = IC$, let the foot of the altitude from I to BC be D. Then $ID = r = [ABC]/s = \cdot 5 \cdot 8 \cdot \sin(60°)/(5+8+7) = \sqrt{3}$. To compute CD, we can use $CD = \frac{1}{2}(BC + AC - AB) = 5$ (consider the incircle and its points of tangency with the triangle). Now Pythagorean gives us $CI = \sqrt{ID^2 + CD^2} = 2\sqrt{7}$.

To compute $IQ = IO + OQ = IO + R$, we first have $R = [ABC]/(4abc) = \frac{7}{\sqrt{3}}$. To compute IO, let the foot of the altitude from O to BC is M, the midpoint of BC, and $OM = \frac{1}{2} \cdot R = \frac{73}{6}$. Then $IOMD$ is a right trapezoid and $IO^2 = (OM - ID)^2 + DM^2 = \left(\frac{\sqrt{3}}{6}\right)^2 + \left(\frac{3}{2}\right)^2 = \frac{7}{3} \Rightarrow IO = \frac{\sqrt{7}}{\sqrt{3}}$. Then finally we have $[XICQ] = \frac{1}{2} \cdot 2\sqrt{7} \cdot \left(\frac{7}{\sqrt{3}} + \frac{\sqrt{7}}{\sqrt{3}}\right) = \frac{7\sqrt{21}+7\sqrt{3}}{3}$, giving us $a+b+c+d+e = \boxed{41}$. □

26. A magic 3×5 board can toggle its cells between black and white. Define a *pattern* to be an assignment of black or white to each of the board's 15 cells (so there are 2^{15} patterns total). Every day after Day 1, at the beginning of the day, the board gets bored with its black-white pattern and makes a new one. However, the board always wants to be unique and will die if any two of its patterns are less than 3 cells different from each other. Furthermore, the board dies if it becomes all white. If the board begins with all cells black on Day 1, compute the maximum number of days it can stay alive.

    Proposed by Janabel Xia

    Solution. $\boxed{2047}$

    To simplify, let's treat the 3×5 board as a vector in $\{0,1\}^{15}$, where 0 is black and 1 is white.

    Define the *distance* between two vectors to be the number of entries that differ. Define a *used* vector to be a vector corresponding to a pattern the board actually makes. Our problem requires a minimum distance of 3 between any two used vectors. This motivates us to consider sets consisting of a vector v itself and the set of vectors of distance 1 from v (you can picture this as the region enclosed by a "radius" of 1 from v), since ensuring no two used vectors overlap in these sets is both necessary and sufficient for the problem's distance 3 condition. We have 2^{15} possible vectors and each used vector eliminates $1 + 15 = 16$ total vectors (including itself), so the board lives at most $2^{15}/16 = 2048$ days by creating a perfect partitioning of the 2^{15} vectors total.

    To prove that on at least one of the 2048 days the board must be all white given that it starts all black on Day 1, we first define the useful notion of the *weight* of a vector to be the number of 1s it has. Let $w(i)$ denote the set of vectors with weight i. We know that to achieve 2048 days, all the used vectors must eliminate each of 2^{15} possible vectors *exactly* once.

    We can calculate in a process as follows: Day 1 (the 0 vector) eliminates all vectors in $w(1)$, and furthermore no vectors in $w(2)$ can be used, so all vectors in $w(2)$ must be eliminated by vectors in $w(3)$. Each vector in $w(3)$ eliminates 3 vectors in $w(2)$, so we must use $\frac{\binom{15}{2}}{3} = 35$ vectors in $w(3)$. Note that each $w(3)$ vector eliminates $15-3 = 12$ vectors in $w(4)$, which we must keep track of. Then we still have $\binom{15}{3} - 35 = 420$ unused vectors in $w(3)$ that must be eliminated entirely by vectors in $w(4)$ (since there can't be any used vectors in $w(2)$). If we continue this fairly straightforward process and keep track of eliminating each vector exactly once, we find that there must be a used vector in $w(15)$, the all white pattern, so $2048 - 1 = \boxed{2047}$ is our final answer. □

27. Let $S_n = \sum_{k=1}^{n}(k^5 + k^7)$. Let the prime factorization of $\gcd(S_{2020}, S_{6060})$ be $p_1^{k_1} \cdot p_2^{k_2} \cdots p_i^{k_i}$. Compute $p_1 + p_2 + \cdots + p_i + k_1 + k_2 + \cdots + k_i$.

    Proposed by Sooyoung Choi

    Solution. $\boxed{121}$

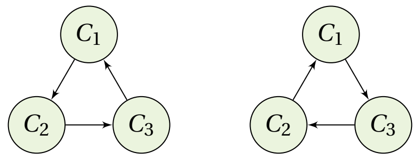
    We claim that $S_n = 2(\frac{n(n+1)}{2})^4$. We use following algebraic manipulation:

$$\sum_{k=1}^{n}(k^5 + k^7) = \frac{1}{8}\sum_{k=1}^{n} 4k^3(2k^4 + 2k^2) = \sum_{k=1}^{n}\{(k^2+k)^2 - (k^2-k)^2\}\{(k^2+k)^2 + (k^2-k)^2\}$$

$$= \frac{1}{8}\sum_{k=1}^{n}[(k^2+k)^4 - (k^2-k)^4] = 2\sum_{k=1}^{n}[(\frac{k(k+1)}{2})^4 - (\frac{(k-1)k}{2})^4] = 2(\frac{n(n+1)}{2})^4$$

    From the expression, we can deduce that $\gcd(S_n, S_{3n}) = \frac{1}{8}n^4$ when $n \equiv 4 \pmod 6$. Therefore, our gcd is $\frac{2020^4}{8} = 2 \cdot 1010^4 = 2^5 \cdot 5^4 \cdot 101^4$, giving $2 + 5 + 101 + 5 + 4 + 4 = \boxed{121}$ as our answer. □

28. A particular country has seven distinct cities, conveniently named $C_1, C_2, \dots, C_7$. Between each pair of cities, a direction is chosen, and a one-way road is constructed in that direction connecting the two cities. After the construction is complete, it is found that any city is reachable from any other city, that is, for distinct $1 \le i, j \le 7$, there is a path of



one-way roads leading from C_i to C_j. Compute the number of ways the roads could have been configured. Pictured on the following page are the possible configurations possible in a country with three cities, if every city is reachable from every other city.

Proposed by Ezra Erives

Solution. $\boxed{1677488}$

We construct a general recursion for $n > 1$ cities. Let the (directed) graph of the n cities be G, which is assumed to have an edge between every pair of vertices, and call a subset S of the vertices *good* if there is no edge from S to G/S and the induced subgraph of G corresponding to S is strongly connected.

We claim that if G is not strongly connected, there there exists some non-empty good subset of G. To see that this is true, note that if G is not strongly connected, there exist $a, b \in G$ for which there exists a path from a to b, but no such path from b to a. Let S_b be the set of vertices reachable from b, including b. If b is reachable from every vertex of S_b, then the induced subgraph corresponding to S_b is strongly connected. Note that any edge from S_b to G/S_b would contradict S_b containing every vertex reachable from b, and thus S_b is good. Else, if S_b is not strongly connected, then there exists some $b' \in S_b$ for which b is not reachable from b'. Set $b = b'$ and repeat the process. As the size of G is finite, the process must eventually halt, at which point we will obtain our good subset.

It follows from our claim that either G is strongly connected, or that there exists some good subset of the vertices. Suppose there were multiple such good subsets for some G. Then these two subsets must be disjoint, and a contradiction would follow by considering the edge between them. Thus there is always at most one good subset.

Let T_n be the number of strongly connected directed graphs with an edge between every pair of vertices (essentially the problem statement, but for general n). To construct a recursion for T_n, we note that if S is some good subset of G of size k, then the induced subgraph corresponding to S satisfies the problem constraints for $n = k$. We then calculate the complement of T_n by enumerating over the size of the good subset in the following manner:

$$2^{\binom{n}{2}} - T_n = \sum_{k=1}^{n-1} \binom{n}{k} T_k 2^{\binom{n-k}{2}}.$$

Starting small, it's easy to see that $T_1 = 1$, $T_2 = 0$, and $T_3 = 2$ (as shown in the example). With the help of a *standard four-function calculator*, or with sufficient computational fortitude, one will eventually arrive at the correct answer of $T_7 = \boxed{1677488}$.

Note: This problem is more formally known as asking for the *number of strongly connected labeled tournaments on n nodes*. If you're curious, check out OEIS entry A054946. □

29. Let $\mathscr{F}$ be the set of polynomials $f(x)$ with integer coefficients for which there exists an integer root of the equation $f(x) = 1$. For all $k > 1$, let m_k be the smallest integer greater than one for which there exists $f(x) \in \mathscr{F}$ such that $f(x) = m_k$ has exactly k distinct integer roots. If the value of $\sqrt{m_{2021} - m_{2020}}$ can be written as $m\sqrt{n}$ for positive integers m, n where n is squarefree, compute the largest integer value of k such that 2^k divides $\frac{m}{n}$.

Proposed by Sooyoung Choi

Solution. $\boxed{1002}$

Assume that for some polynomial $f \in \mathscr{F}$, $f(x) = m_k$ has k distinct integer roots $\beta_1, \beta_2, \cdots, \beta_k$. Then, there exists a polynomial $g(x)$ with integer coefficients such that $f(x) - m_k = (x-\beta_1)(x-\beta_2)\cdots(x-\beta_k)g(x)$. There also exists an integer α such that $f(\alpha) = 1$, since $f \in \mathscr{F}$.

If we plug α in the above equation with $g(x)$ and take the absolute value, we have

$$m_k - 1 = |\alpha - \beta_1| \times |\alpha - \beta_2| \times \cdots \times |\alpha - \beta_k| \times |g(\alpha)|$$

Now from the problem condition, we know that $\alpha - \beta_i$ are distinct and nonzero since $m_k > 1$. Therefore, to minimize m_k, we have $|g(\alpha)| = 1$ and $|\alpha - \beta_i$ will take the smallest k distinct nonzero integers. Thus, when k is even, we have

$$m_k = [\left(\frac{k}{2}\right)!]^2 + 1$$

and when k is odd, we have

$$m_k = \left(\frac{k-1}{2}\right)!\left(\frac{k+1}{2}\right)! + 1.$$

Hence, $m_{2021} - m_{2020} = 1011!1010! - 1010!1010! = 1010 \cdot 1010!^2$. Computing gives us $\boxed{1002}$ as our final answer.

□

30. Let $ABCD$ be a cyclic quadrilateral such that the ratio of its diagonals is $AC : BD = 7 : 5$. Let E and F be the intersections of lines AB and CD and lines BC and AD, respectively. Let L and M be the midpoints of diagonals AC and BD, respectively. Given that $EF = 2020$, the length of LM can be written as $\frac{p}{q}$ where p, q are relatively prime positive integers. Compute $p + q$.

Proposed by Sooyoung Choi

Solution. $\boxed{4855}$

We claim that the following equation is true:

$$\frac{LM}{EF} = \frac{1}{2}\left(\frac{AC}{BD} - \frac{BD}{AC}\right)$$

We will use vectors to prove the result. Let i, j be the unit vector from origin E and direction $\overrightarrow{EB}, \overrightarrow{EC}$, respectively. Since $ABCD$ is cyclic, we have $EB \cdot EA = EC \cdot ED$. Thus, we can find some constant $\mu > \lambda$ and k such that

$$\overrightarrow{EB} = i, \overrightarrow{EA} = k\mu i, \overrightarrow{EC} = \mu j, \overrightarrow{ED} = k\lambda j.$$

Then,

$$\overrightarrow{LM} = \overrightarrow{EM} - \overrightarrow{EL} = \frac{1}{2}[(\overrightarrow{EB} + \overrightarrow{ED}) - (\overrightarrow{EA} + \overrightarrow{EC})] = \frac{1}{2}[(\lambda - k\mu)i + (k\lambda - \mu)j].$$

Also, we have

$$\overrightarrow{AC}^2 = |\overrightarrow{EC} - \overrightarrow{EA}|^2 = \mu^2|j - ki|^2 = \mu^2(k^2 + 1 - 2k\cos\alpha)$$
$$\overrightarrow{BD}^2 = |\overrightarrow{ED} - \overrightarrow{EB}|^2 = \lambda^2|kj - i|^2 = \lambda^2(k^2 + 1 - 2k\cos\alpha)$$

where α is the angle formed by lines EB and EC.

On the other hand, since F lies on both AD and BC, we can write $\overrightarrow{EF}$ in the following two ways for some scalars t, s:

$$\overrightarrow{EF} = t\overrightarrow{EA} + (1-t)\overrightarrow{ED} = tk\mu i + (1-t)k\lambda j,$$
$$\overrightarrow{EF} = s\overrightarrow{EB} + (1-s)\overrightarrow{EC} = s\lambda i + (1-s)\mu j.$$

Hence, we have $s = tk\mu$ and $(1-s)\mu = (1-t)k\mu$. Solving this system of equations, we get $s = \frac{\mu(\mu - k\lambda)}{\mu^2 - \lambda^2}$. Plugging s back into the expression of $\overrightarrow{EF}$, we have

$$\overrightarrow{EF} = \frac{\lambda\mu}{\mu^2 - \lambda^2}[(\mu - k\lambda)i + (k\mu - \lambda)j]$$

Finally, we see that

$$\frac{LM^2}{EF^2} = \frac{1}{4}\left(\frac{\mu^2-\lambda^2}{\lambda\mu}\right)^2 = \frac{1}{4}\left(\frac{\mu}{\lambda}-\frac{\lambda}{\mu}\right)^2 = \frac{1}{4}\left(\frac{AC}{BD}-\frac{BD}{AC}\right)^2$$

and square rooting both sides proves our claim.

Using our claim, we have $LM = EF \cdot \frac{1}{2}(\frac{7}{5}-\frac{5}{7}) = \frac{4848}{7}$, giving us $4848+7 = \boxed{4855}$ as our final answer.

□