LMT Spring Online

May 9th – May 15th, 2020

Contest Instructions

Contest Window

The competition consists of a single round, consisting of 30 short answer problems. All answers are non-negative integers. The problems will be made available on the homepage of the LMT website on **Saturday, May 9th, at 12:00 pm**. Teams will have until **Friday, May 15th, at 3:00 pm** to submit their answers using the link provided by email.

Contest Rules

With the exception of **standard four-function calculators**, computational aids including but not limited to scientific and graphing calculators, computer programs, and software such as Geogebra, Mathematica, and WolframAlpha, are **not** allowed. Communication of any form between students on different teams is similarly prohibited, and any team caught either giving or receiving an unfair advantage over other competitors will be disqualified. What constitutes cheating will be up to the final discretion of the competition organizers, who reserve the right to disqualify any team suspected of violating these rules.

Submitting Answers and Editing Team Information

During registration, your captain will be emailed a link to your team's homepage. This is where you will be able to update team information and answer submissions. We recommend that your captain distribute this link to the rest of the team so that the entire team has access. Once on your team's homepage, to enter or edit team answers, click the link next to "Submission Link:". Team name, team member names, and grades may be edited through the homepage as well. Remember, team member names must be the real names of the people on your team, and the team name must be appropriate.

Errata

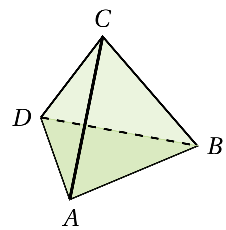
If you believe there to be an error in one of the questions, email us at lmt@lhsmath.org with "Clarification" as the subject. Clarifications for problems will be updated on the LMT homepage, if necessary.

Scoring

The score of your team is the number of questions you answer correctly. We will break ties by weighing the problems based on how many teams solve them. Results will be posted shortly after the competition, where the top teams will be recognized.

1. Compute the smallest nonnegative integer that can be written as the sum of 2020 distinct integers.

2. In tetrahedron $ABCD$, as shown below, compute the number of ways to start at A, walk along some path of edges, and arrive back at A without walking over the same edge twice.



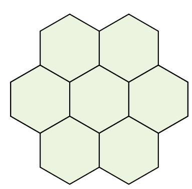
3. Let LMT represent a 3-digit positive integer where L and M are nonzero digits. Suppose that the 2-digit number MT divides LMT. Compute the difference between the maximum and minimum possible values of LMT.

4. Suppose there are n ordered pairs of positive integers (a_i, b_i) such that $a_i + b_i = 2020$ and $a_i b_i$ is a multiple of 2020, where $1 \le i \le n$. Compute the sum
$$\sum_{i=1}^{n} a_i + b_i.$$

5. For a positive integer n, let $\mathscr{D}(n)$ be the value obtained by, starting from the left, alternating between adding and subtracting the digits of n. For example, $\mathscr{D}(321) = 3 - 2 + 1 = 2$, while $\mathscr{D}(40) = 4 - 0 = 4$. Compute the value of the sum
$$\sum_{n=1}^{100} \mathscr{D}(n) = \mathscr{D}(1) + \mathscr{D}(2) + \cdots + \mathscr{D}(100).$$

6. Let $\triangle ABC$ be a triangle such that $AB = 6, BC = 8$, and $AC = 10$. Let M be the midpoint of BC. Circle ω passes through A and is tangent to BC at M. Suppose ω intersects segments AB and AC again at points X and Y, respectively. If the area of AXY can be expressed as $\frac{p}{q}$ where p, q are relatively prime integers, compute $p + q$.

7. The hexagonal pattern constructed below has two smaller hexagons per side and has a total of 30 edges. A similar figure is constructed with 20 smaller hexagons per side. Compute the number of edges in this larger figure.

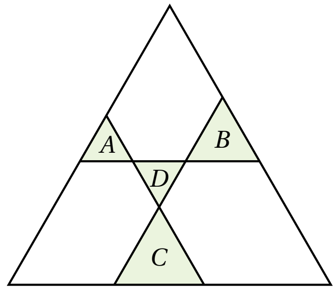


8. Let a, b be real numbers satisfying $a^2 + b^2 = 3ab = 75$ and $a > b$. Compute $a^3 - b^3$.

9. A function $f(x)$ is such that for any integer x, $f(x) + xf(2-x) = 6$. Compute $-2019f(2020)$.

10. Three mutually externally tangent circles are internally tangent to a circle with radius 1. If two of the inner circles have radius $\frac{1}{3}$, the largest possible radius of the third inner circle can be expressed in the form $\frac{a+b\sqrt{c}}{d}$ where c is squarefree and $\gcd(a, b, d) = 1$. Find $a + b + c + d$.

11. Let set $\mathscr{S}$ contain all positive integers less than or equal to 2020 that can be written in the form $n(n+1)$ for some positive integer n. Compute the number of ordered pairs (a, b) such that $a, b \in \mathscr{S}$ and $a - b$ is a power of two.



12. In the figure above, the large triangle and all four shaded triangles are equilateral. If the areas of triangles A, B, and C are 1, 2, and 3, respectively, compute the smallest possible integer ratio between the area of the entire triangle to the area of triangle D.

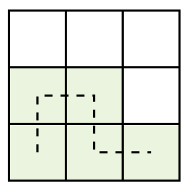
13. In the game of *Flow*, a path is drawn through a 3×3 grid of squares obeying the following rules:

    i A path is continuous with no breaks (it can be drawn without lifting a pencil).

    ii A path that spans multiple squares can only be drawn between colored squares that share a side.

    iii A path cannot go through a square more than once.

    Compute the number of ways to color a positive number of squares on the grid such that a valid path can be drawn. An example of one such coloring and a valid path is shown below.



14. Let $\triangle ABC$ be a triangle such that $AB = 40$ and $AC = 30$. Points X and Y are on the segment AB and BC, respectively such that $AX : BX = 3:2$ and $BY : CY = 1:4$. Given that $XY = 12$, the area of $\triangle ABC$ can be written as $a\sqrt{b}$ where a and b are positive integers and b is squarefree. Compute $a+b$.

15. Let $\phi(k)$ denote the number of positive integers less than or equal to k that are relatively prime to k. For example, $\phi(2) = 1$ and $\phi(10) = 4$. Compute the number of positive integers $n \le 2020$ such that $\phi(n^2) = 2\phi(n)^2$.

16. For non-negative integer n, the function f is given by

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } n \text{ is even} \\ x-1 & \text{if } n \text{ is odd.} \end{cases}$$

    Furthermore, let $h(n)$ be the smallest k for which $f^k(n) = 0$. Compute

$$\sum_{n=1}^{1024} h(n).$$

17. Let ABC be a triangle such that $AB = 26$, $AC = 30$, and $BC = 28$. Let C' and B' be the reflections of the circumcenter O over AB and AC, respectively. The length of the portion of line segment $B'C'$ inside triangle ABC can be written as $\frac{p}{q}$, where p, q are relatively prime positive integers. Compute $p+q$.

18. Compute the maximum integer value of k such that 2^k divides $3^{2n+3} + 40n - 27$ for any positive integer n.

19. Let ABC be a triangle such that such that $AB = 14, BC = 13$, and $AC = 15$. Let X be a point inside triangle ABC. Compute the minimum possible value of $(\sqrt{2}AX + BX + CX)^2$.

20. Let $c_1 < c_2 < c_3$ be the three smallest positive integer values of c such that the distance between the parabola $y = x^2 + 2020$ and the line $y = cx$ is a rational multiple of $\sqrt{2}$. Compute $c_1 + c_2 + c_3$.

21. Let $\{a_n\}$ be the sequence such that $a_0 = 2019$ and
$$a_n = -\frac{2020}{n}\sum_{k=0}^{n-1} a_k.$$
Compute the last three digits of $\sum_{n=1}^{2020} 2020^n a_n n$.

22. The numbers one through eight are written, in that order, on a chalkboard. A mysterious higher power in possession of both an eraser and a piece of chalk chooses three distinct numbers x, y, and z on the board, and does the following. First, x is erased and replaced with y, after which y is erased and replaced with z, and finally z is erased and replaced with x. The higher power repeats this process some finite number of times. For example, if $(x, y, z) = (2,4,5)$ is chosen, followed by $(x, y, z) = (1,4,3)$, the board would change in the following manner:
$$12345678 \to 14352678 \to 43152678$$
Compute the number of possible final orderings of the eight numbers.

23. Let $\triangle ABC$ be a triangle such that $AB = AC = 40$ and $BC = 79$. Let X and Y be the points on segments AB and AC such that $AX = 5, AY = 25$. Given that P is the intersection of lines XY and BC, compute $PX \cdot PY - PB \cdot PC$.

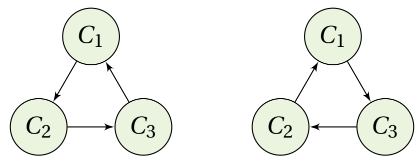
24. Let a, b, and c be real angles such that
$$3\sin a + 4\sin b + 5\sin c = 0$$
$$3\cos a + 4\cos b + 5\cos c = 0.$$
The maximum value of the expression $\frac{\sin b \sin c}{\sin^2 a}$ can be expressed as $\frac{p}{q}$ for relatively prime p, q. Compute $p + q$.

25. Let $\triangle ABC$ be a triangle such that $AB = 5, AC = 8$, and $\angle BAC = 60^\circ$. Let Γ denote the circumcircle of ABC, and let I and O denote the incenter and circumcenter of $\triangle ABC$, respectively. Let P be the intersection of ray IO with Γ, and let X be the intersection of ray BI with Γ. If the area of quadrilateral $XICP$ can be expressed as $\frac{a\sqrt{b}+c\sqrt{d}}{e}$, where a and d are squarefree positive integers and $\gcd(a, c, e) = 1$, compute $a + b + c + d + e$.

26. A magic 3×5 board can toggle its cells between black and white. Define a *pattern* to be an assignment of black or white to each of the board's 15 cells (so there are 2^{15} patterns total). Every day after Day 1, at the beginning of the day, the board gets bored with its black-white pattern and makes a new one. However, the board always wants to be unique and will die if any two of its patterns are less than 3 cells different from each other. Furthermore, the board dies if it becomes all white. If the board begins with all cells black on Day 1, compute the maximum number of days it can stay alive.

27. Let $S_n = \sum_{k=1}^{n}(k^5 + k^7)$. Let the prime factorization of $\gcd(S_{2020}, S_{6060})$ be $p_1^{k_1} \cdot p_2^{k_2} \cdots p_i^{k_i}$. Compute $p_1 + p_2 + \cdots + p_i + k_1 + k_2 + \cdots + k_i$.

28. A particular country has seven distinct cities, conveniently named $C_1, C_2, \ldots, C_7$. Between each pair of cities, a direction is chosen, and a one-way road is constructed in that direction connecting the two cities. After the construction is complete, it is found that any city is reachable from any other city, that is, for distinct $1 \le i, j \le 7$, there is a path of one-way roads leading from C_i to C_j. Compute the number of ways the roads could have been configured. Pictured on the following page are the possible configurations possible in a country with three cities, if every city is reachable from every other city.



29. Let $\mathscr{F}$ be the set of polynomials $f(x)$ with integer coefficients for which there exists an integer root of the equation $f(x) = 1$. For all $k > 1$, let m_k be the smallest integer greater than one for which there exists $f(x) \in \mathscr{F}$ such that $f(x) = m_k$ has exactly k distinct integer roots. If the value of $\sqrt{m_{2021} - m_{2020}}$ can be written as $m\sqrt{n}$ for positive integers m, n where n is squarefree, compute the largest integer value of k such that 2^k divides $\frac{m}{n}$.

30. Let $ABCD$ be a cyclic quadrilateral such that the ratio of its diagonals is $AC : BD = 7 : 5$. Let E and F be the intersections of lines AB and CD and lines BC and AD, respectively. Let L and M be the midpoints of diagonals AC and BD, respectively. Given that $EF = 2020$, the length of LM can be written as $\frac{p}{q}$ where p, q are relatively prime positive integers. Compute $p + q$.