Guts Round Solutions

Lexington High School

December 12th, 2020

10th Annual Lexington Math Tournament - Guts Round - Part 1

Team Name: ______________________________

______________ 1. **[5]** Find the remainder when 2020! is divided by 2020^2.

Proposed by Kevin Zhao

Solution. $\boxed{0}$

$2020! = (2\cdot 1010\cdot 2020)(1\cdot 3\cdot 4\cdots 2019)$. Thus 2020^2 is a factor of 2020!, so the remainder of 2020! when it is divided by 2020^2 is $\boxed{0}$. □

______________ 2. **[5]** In a five term arithmetic sequence, the first term is 2020 and the last term is 4040. Find the second term of the sequence.

Proposed by Ada Tsui

Solution. $\boxed{2525}$

In an arithmetic sequence, the third term is the average of the first and fifth terms, and the second term is the average of the first and third terms. Applying this, the third term is $\frac{2020+4040}{2} = 3030$, and the second term is $\frac{2020+3030}{2} = \boxed{2525}$, which is what we are looking for. □

______________ 3. **[5]** Circles C_1, C_2, and C_3 have radii 2, 3, and 6, and respectively. If the area of a fourth circle C_4 is the sum of the areas of C_1, C_2, and C_3, compute the radius of C_4.

Proposed by Alex Li

Solution. $\boxed{7}$

Since the area of a circle is $A = \pi r^2$, the sum of the areas of the three circles is $4\pi + 9\pi + 36\pi = 49\pi$. Thus the radius of C_4 is $\sqrt{49} = \boxed{7}$. □

10th Annual Lexington Math Tournament - Guts Round - Part 2

Team Name: ______________________________

______________ 4. **[5]** At Lexington High School, each student is given a unique five-character ID consisting of uppercase letters. Compute the number of possible IDs that contain the string "LMT".

Proposed by Alex Li

Solution. $\boxed{2028}$

The string "LMT" must occur at maximum once, because two cannot overlap and there are less than $2\cdot 3$ possible characters. LMT can either occupy the first three, middle three, or last three vertices. For the remaining two characters, there are 26 choices for letters that could occupy those locations. Thus the number of IDs is $3\cdot 26^2 = \boxed{2028}$. □

______ 5. **[5]** For what digit d is the base 9 numeral $7d35_9$ divisible by 8?

Proposed by Alex Li

Solution. $\boxed{1}$

Note that in base 9, a numeral is divisible by 8 if the sum of its digits is divisible by 8. This can be seen by noting that the numeral $\overline{abcd} = 9^3a+9^2b+9c+d \equiv a+b+c+d \pmod 8$. Thus $7+d+3+5 = 15+d$, must be divisible by 8, which means $d = \boxed{1}$. □

______ 6. **[5]** The number 2021 can be written as the sum of 2021 consecutive integers. What is the largest term in the sequence of 2021 consecutive integers?

Proposed by Taiki Aiba

Solution. $\boxed{1011}$

Let the smallest term in the sequence be k. We have the equation

$$2021 = k+(k+1)+(k+2)+\cdots+(k+2020),$$

which can be rewritten as

$$2021 = 2021k+1+2+3+\cdots+2020 = 2021k+\frac{2021(2020)}{2}.$$

Multiplying both sides of the equation by $\frac{2}{2021}$, we get

$$2 = 2k+2020 \implies k = -1009.$$

Thus, the largest element is

$$k+2020 = \boxed{1011}.$$

Alternate Solution by Zachary Perry:

Note that the sum of the numbers in an arithmetic sequence is the median times the number of terms. This implies that the median of this sequence is 1, so the largest term is $\frac{2021-1}{2}+1 = \boxed{1011}$. □

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10th Annual Lexington Math Tournament - Guts Round - Part 3

Team Name: ______________________

______ 7. **[6]** $2020\cdot N$ is a perfect integer cube. If N can be expressed as $2^a\cdot5^b\cdot101^c$, find the least possible value of $a+b+c$ such that a, b, c, are all positive integers and not necessarily distinct.

Proposed by Ephram Chun

Solution. $\boxed{5}$

A number is a perfect cube when its prime factors each occur a multiple of three times. Since $2020 = 2^2*5*101$, we must have 2, 5 and 101 in our prime factorization, and to minimize N we wish to minimize the number of times those occur, so we let each occur 3 times and no other factors occur. Thus we want $a = 1, b = 2, c = 2$. Therefore our answer is

$$1+2+2 = \boxed{5}.$$

□

______ 8. **[6]** A rhombus with side length 1 has an inscribed circle with radius $\frac{1}{3}$. If the area of the rhombus can be expressed as $\frac{a}{b}$ for relatively prime, positive a, b, evaluate $a+b$.

Proposed by Alex Li

Solution. $\boxed{5}$

Recall that a line tangent to a circle is perpendicular to the radius of the circle at the tangent point. Since a rhombus is also a parallelogram, the diameter of the inscribed circle will be perpendicular to opposite sides of the rhombus. This means the height of the rhombus is equal to the diameter of the circle, or $\frac{2}{3}$. Since the base of the rhombus is 1, its area is $A = bh = \frac{2}{3} \cdot 1 = \frac{2}{3}$. The answer is $2+3 = \boxed{5}$. □

______________ 9. **[6]** If $xy : yz : zx = 6 : 8 : 12$, and $x^3 + y^3 + z^3 : xyz$ is $m : n$ where m and n are relatively prime positive integers, then find $m+n$.

Proposed by Ada Tsui

Solution. $\boxed{41}$

Let $xy = 6k^2, yz = 8k^2, zx = 12k^2$. (Note that this corresponds with the ratio given in the problem.) Then multiplying all of the equations together gives $x^2y^2z^2 = 576k^6$, and $xyz = 24k^3$. Dividing that equation by each of our equations, we get $x = 3k, y = 2k, z = 4k$.

Plugging our values for x, y, z into our second ratio, we get

$$(3k)^3 + (2k)^3 + (4k)^3 : (3k)(2k)(4k)$$

$$99k^3 : 24k^3 = 33 : 8,$$

so $m = 33, n = 8$, and $m+n = 33+8 = \boxed{41}$. □

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Team Name: ______________________________

______________ 10. **[6]** 2020 magicians are divided into groups of 2 for the Lexington Magic Tournament. After every 5 days, which is the duration of one match, teams are rearranged so no 2 people are ever on the same team. If the longest tournament is n days long, what is the value of n?

Proposed by Ephram Chun

Solution. $\boxed{10095}$

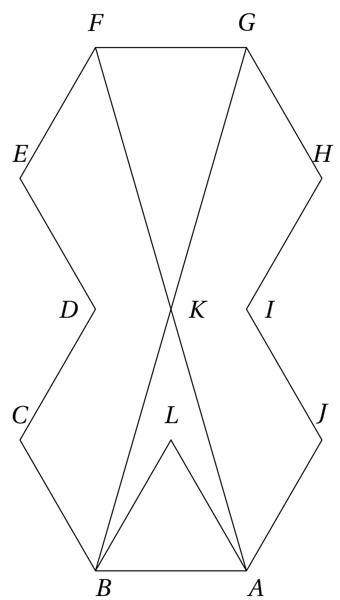
We can see that there are 2019 ways to pair with at least 2 other magicians. Therefore the longest tournament will be $5 * 2019 = \boxed{10095}$. We can prove that there are 2019 ways to pair with a different magician since if we have a circle with a magician in the center. We can connect the magician in the center of the circle to a magician on the circle and all other pairs are made "perpendicular" to that radius. □

______________ 11. **[6]** Cai and Lai are eating cookies. Their cookies are in the shape of two regular hexagons glued to each other, and the cookies have area 18 square units. They each make a cut along the two long diagonals of a cookie; this now makes four pieces for them to eat and enjoy. What is the minimum area among the four pieces?

Proposed by Richard Chen

Solution. $\boxed{3}$

We use the fact that the area of a triangle is half of the base times height.



Note that triangle ABK shares base AB with ABL, and has twice the height of ABL. Note that triangle ABL has area $\frac{1}{12}$ of the whole cookie, since we can split up the cookie into twelve congruent equilateral triangles. Thus, ABK is twice the area of ABL and is $\frac{1}{6}$ the area of the whole cookie, or $\frac{1}{6} \cdot 18 = \boxed{3}$. □

______________ 12. **[6]** If the value of the infinite sum

$$\frac{1}{2^2-1^2}+\frac{1}{4^2-2^2}+\frac{1}{8^2-4^2}+\frac{1}{16^2-8^2}+\cdots.$$

can be expressed as $\frac{a}{b}$ for relatively prime positive integers a, b, evaluate $a+b$.

Proposed by Alex Li

Solution. $\boxed{13}$

This summation can be written as

$$S = \sum_{k=0}^{\infty} \frac{1}{(2^{k+1})^2 - (2^k)^2}.$$

Factoring the denominator gives

$$\begin{aligned} S &= \sum_{k=0}^{\infty} \frac{1}{(2^{k+1}+2^k)(2^{k+1}-2^k)} \\ &= \sum_{k=0}^{\infty} \frac{1}{(2^{k+1}+2^k)(2^k)} \\ &= \sum_{k=0}^{\infty} \frac{1}{(2+1)(2^k)(2^k)} \\ &= \frac{1}{3}\sum_{k=0}^{\infty} \frac{1}{4^k} \\ &= \frac{1}{3}\cdot\frac{4}{3} = \frac{4}{9} \end{aligned}$$

The answer is $4+9 = \boxed{13}$. □

10th Annual Lexington Math Tournament - Guts Round - Part 5

Team Name: ________________________________

________________ 13. **[7]** Let set S contain all positive integers that are one less than a perfect square. Find the sum of all powers of 2 that can be expressed as the product of two (not necessarily distinct) members of S.

Proposed by Alex Li

Solution. $\boxed{64}$

If the product of two members of set S is a power of two, then the two members themselves must be powers of two. Thus we wish to find all n such that n^2-1 is a power of two. Since $n^2-1 = (n+1)(n-1)$, both $n+1$ and $n-1$ must be powers of two. The only powers of two that differ by 2 are 4 and 2, which means that we must have $n = 3$ and the only member of set S is 8. Thus the only power of two is $\boxed{64}$. □

________________ 14. **[7]** Ada and Emily are playing a game that ends when either player wins, after some number of rounds. Each round, either nobody wins, Ada wins, or Emily wins. The probability that neither player wins each round is $\frac{1}{5}$ and the probability that Emily wins the game as a whole is $\frac{3}{4}$. If the probability that in a given round Emily wins is $\frac{m}{n}$ such that m and n are relatively prime integers, then find $m+n$.

Proposed by Ada Tsui

Solution. $\boxed{8}$

Let the probability that Emily wins a given round be p, so the probability that Ada wins a given round is $\frac{4}{5} - p$.

The ratio of the probability that Emily wins a given round to the probability that Ada wins a given round is the same as the ratio of the probability that Emily wins the whole game to the probability that Ada wins the whole game are the same. The latter ratio is $\frac{3}{4} : \left(1-\frac{3}{4}\right) = \frac{3}{4} : \frac{1}{4} = 3:1$.

Allowing the former ratio to equal the latter ratio, we get $p : \left(\frac{4}{5} - p\right) = 3:1$, or $\frac{3}{5}$. $\left(p = \frac{12}{5} - 3p, 4p = \frac{12}{5}.\right)$

Thus, $m = 3, n = 5$, and $m+n = \boxed{8}$. □

______________ 15. **[7]** $\triangle ABC$ has $AB = 5$, $BC = 6$, and $AC = 7$. Let M be the midpoint of BC, and let the circumcircle of $\triangle ABM$ intersect AC at N. If the length of segment MN can be expressed as $\frac{a}{b}$ for relatively prime positive integers a, b, find $a + b$.

Proposed by Alex Li

Solution. $\boxed{22}$

Note that $\angle MNC = \angle B$ by cyclic quad properties. Then $\triangle CNM \sim \triangle CBA$ because $\angle C = \angle C$ and the previous angle property. Thus

$$MN = AB \cdot \frac{CM}{CA} = 5 \cdot \frac{3}{7} = \frac{15}{7}.$$

Thus the wanted answer is $7 + 15 = \boxed{22}$. □

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10th Annual Lexington Math Tournament - Guts Round - Part 6

Team Name: ____________________________

______________ 16. **[7]** Compute

$$\frac{2019! \cdot 2^{2019}}{(2020^2 - 2018^2)(2020^2 - 2016^2) \cdots (2020^2 - 2^2)}.$$

Proposed by Ada Tsui

Solution. $\boxed{2020}$

Use the difference of squares to expand the denominator to get

$$(2020^2 - 2018^2)(2020^2 - 2016^2) \cdots (2020^2 - 2^2) =$$

$$(2020 - 2018)(2020 + 2018)(2020 - 2016)(2020 + 2016) \cdots (2020 - 2)(2020 + 2),$$

and factor out 2 from each factor to get

$$2^{2018}(1)(2019)(2)(2018) \cdots (1009)(1011) = \frac{2^{2018} 2019!}{1010}.$$

Plug that back into the fraction to get

$$\frac{2019! \cdot 2^{2019}}{\frac{2019! \cdot 2^{2018}}{1010}} = 2 \cdot 1010 = \boxed{2020}.$$

□

______________ 17. **[7]** In a regular square room of side length $2\sqrt{2}$ ft, two cats that can see 2 feet ahead of them are randomly placed into the four corners such that they do not share the same corner. If the probability that they don't see the mouse, also placed randomly into the room, can be expressed as $\frac{a-b\pi}{c}$, where a, b, c are positive integers with a greatest common factor of 1, then find $a + b + c$.

Proposed by Ada Tsui

Solution. $\boxed{12}$

We approach this problem with casework on the positions of the cats and complementary counting.

There are two cases based on the positions of the cats: the cats are adjacent to each other, and the cats are opposite of each other.

The probability of the first case is $\frac{2}{3}$, and the probability of the second case is $\frac{1}{3}$ (Once the first corner is chosen, two of the remaining corners are adjacent to that corner, and one of the remaining corners is opposite of that corner.)

In both cases, the total area the mouse could be in is $2\sqrt{2}\cdot 2\sqrt{2}=8$.

In the first case, the area of the cat's vision is two $45°$ arcs of radius 2 and one right isosceles triangle of leg length 2, or $2\cdot\frac{45}{360}\cdot 2^2\cdot\pi+\frac{2\cdot 2}{2}=\pi+2$, and the probability that the mouse is in the cat's vision is $\frac{\pi+2}{8}$.

In the second case, the area of the cat's vision is two $90°$ arcs of radius 2, or $2\cdot\frac{90}{360}\cdot 2^2\cdot\pi=2\pi$ (note that the two circles do not intersect), and the probability that the mouse is in the cat's vision is $\frac{2\pi}{8}$.

Bringing it all together, the probability that the mouse is in the cat's vision is $\frac{2}{3}\cdot\frac{\pi+2}{8}+\frac{1}{3}\cdot\frac{2\pi}{8}=\frac{4\pi+4}{24}=\frac{\pi+1}{6}$.

Of course, we are looking for the probability that the mouse is not in the cat's vision, which is just $1-\frac{\pi+1}{6}=\frac{5-\pi}{6}$.

Thus, $a=5, b=1, c=6$, and $a+b+c=5+1+6=\boxed{12}$. □

18. **[7]** Given that $\sqrt{x+2y}-\sqrt{x-2y}=2$, compute the minimum value of $x+y$.

Proposed by Alex Li

Solution. $\boxed{3}$

Adding $\sqrt{x-2y}$ to both sides and squaring, the given equation can be simplified:

$$\begin{aligned} &x+2y=x-2y+4+4\sqrt{x-2y}\\ \Rightarrow &y-1=\sqrt{x-2y}\\ \Rightarrow &y^2+1=x \end{aligned}$$

Plugging in this expression for x into the original equation, we obtain the equation $|y+1|-|y-1|=2$, which is true if and only if $y\geq 1$. Thus the minimum possible value of $x+y$ is $1+2=\boxed{3}$. □

10th Annual Lexington Math Tournament - Guts Round - Part 7

Team Name: ______________________________

______________ 19. **[8]** Find the second smallest prime factor of $18!+1$.

Proposed by Kaylee Ji

Solution. $\boxed{23}$

We claim the answer is 23.

For any $p \le 18$, $p \mid 18!$ so p does not divide $18!+1$.

Now we consider primes larger than 18.

By Wilson's Theorem, $18! \equiv -1 \pmod{19}$. Therefore, 19 divides $18!+1$.

Now, $18! \equiv 24 \cdot 18! \equiv (-1)(-2)(-3)(-4)18! \equiv 22 \cdot 21 \cdot 20 \cdot 19 \cdot 18! \equiv 22! \equiv -1 \pmod{23}$ so 23 also divides $18!+1$.

Since 19 is the only prime less than 23 that divides $18!+1$, $\boxed{23}$ is the second smallest prime factor of $18!+1$. □

______________ 20. **[8]** Cyclic quadrilateral $ABCD$ has $AC = AD = 5, CD = 6$, and $AB = BC$. If the length of AB can be expressed as $\frac{a\sqrt{b}}{c}$ where a, c are relatively prime positive integers and b is square-free, evaluate $a+b+c$.

Proposed by Ada Tsui

Solution. $\boxed{14}$

Let $\angle BAC = \angle BCA = x^\circ$, so $\angle B = 180-2x^\circ$. Since $ABCD$ is cyclic, $\angle ADC = 2x^\circ$.

Dropping an altitude from A to CD creates a 3-4-5 triangle with the altitude, CD, and the segment from D to the foot of the altitude that allows us to see $\cos D = \cos(2x) = \frac{3}{5}$.

The half-angle Trigonometric identity yields $\sin x = \sqrt{\frac{1-\cos(2x)}{2}} = \sqrt{\frac{1-\frac{3}{5}}{2}} = \sqrt{\frac{1}{5}}$, and the Trigonometric identity $\sin^2 x + \cos^2 x = 1$ yields $\cos = \sqrt{\frac{4}{5}} = \frac{2\sqrt{5}}{5}$.

Dropping an altitude from B to AC creates a triangle with the altitude, AB, and the segment from A to the foot of the altitude with the x° angle and AB, which is what we are looking for.

From here, all we need to do is apply $\cos = \frac{2\sqrt{5}}{5}$ to the triangle we just created to get $\frac{2\sqrt{5}}{5} = \frac{\frac{5}{2}}{AB}$. Solving gives $AB = \frac{5\sqrt{5}}{4}$.

Thus, $a = 5, b = 5, c = 4$, and $a+b+c = 5+5+4 = \boxed{14}$. □

______________ 21. **[8]** A sequence with first term a_0 is defined such that $a_{n+1} = 2a_n^2 - 1$ for $n \ge 0$. Let N denote the number of possible values of a_0 such that $a_0 = a_{2020}$. Find the number of factors of N.

Proposed by Alex Li

Solution. $\boxed{2021}$

Letting $a_0 = \cos\theta$, we see that $a_1 = \cos 2\theta$, $a_2 = \cos 4\theta$, and more generally. $a_n = \cos 2^n\theta$. We thus wish to find the number of distinct values of $\cos\theta$ given that $\cos\theta = \cos 2^{2020}\theta$.

Graphing the two cosine functions on the interval $[0,\pi]$, we see that $y = \cos 2^{2020}\theta$ contains 2^{2020} inflection points. Each inflection point corresponds to one intersection with the graph $y = \cos\theta$, which means there are 2^{2020} intersection points. Moreover, since $y = \cos\theta$ is monotonically decreasing on the interval $[0,\pi]$, each of these values are distinct. Thus $N = 2^{2020}$, and the answer is $\boxed{2021}$. □

10th Annual Lexington Math Tournament - Guts Round - Part 8

Team Name: ______________________________

______________ 22. **[8]** Find the area of a triangle with side lengths $\sqrt{13}$, $\sqrt{29}$, and $\sqrt{34}$. The area can be expressed as $\frac{m}{n}$ for m,n relatively prime positive integers, then find $m+n$.

Proposed by Kaylee Ji

Solution. $\boxed{21}$

Notice this triangle can be inscribed in a 5 by 5 square since $3^2+5^2=34$, $2^2+5^2=29$, and $3^2+2^2=13$.

The area of the triangle is the area of the square minus the area of the three right triangles bounding our triangle. Therefore the area of the triangle is $5\cdot5-\frac{1}{2}(3\cdot2+3\cdot5+5\cdot3)=\frac{19}{2}$. $19+2=\boxed{21}$. □

______________ 23. **[8]** Let $f:\mathbb{R}\backslash 0\to\mathbb{R}\backslash 0$ be a non-constant, continuous function defined such that $f(3^x2^y)=\frac{y}{x}f(2^x)+\frac{x}{y}f(3^y)$ for any $x,y\neq0$. Compute $\frac{f(1296)}{f(6)}$.

Proposed by Richard Chen and Zachary Perry

Solution. $\boxed{4}$

□

______________ 24. **[8]** In the Oxtingnle math team, there are 5 students, numbered 1 to 5, all of which either always tell the truth or always lie. When Marpeh asks the team about how they did in a 10 question competition, each student i makes 5 separate statements (so either they are all false or all true): "I got problems $i+1$ to $2i$, inclusive, wrong", and then "Student j got both problems i and $2i$ correct" for all $j\neq i$. What is the most problems the team could have gotten correctly?

Proposed by Jeff Lin

Solution. $\boxed{38}$

If student 1 is telling the truth, then all other students got 1 and 2 correct, while 1 got 2 incorrect. That means that 2 must be lying, so no student got both 2 and 4 correct (other than 2). Since 5 got 2 correct, he got 4 incorrect, which must mean 4 is lying as well. Thus, nobody got both 4 and 8. For 3, telling the truth would mean 8 questions correct for their teammates (4 that they , while making them get 3 wrong (one that they would get wrong anyways), so that would result in a better score. On the other hand, if student 1 is lying, then they got 2 correct, and no other student got both 1 and 2 correct. Assuming they all got 2 correct, 2 could be telling the truth, allowing everyone to get both 2 and 4. However, that would mean 2 gets 3 and 4 wrong, which means that neither 3 nor 4 can be telling the truth, making it so that all the teammates get 6 and 8 wrong, which is worse. For 5, however, lying is actually better, because he can get $6,7,9$, and 10 while only making 3 people miss 10 (3 gets 5 wrong anyways). We can also assume that everyone solves 5, 7, and 9, because there are no restraints there. This gives us $1011111010, 1111111010, 1110001111, 1110111110$, and 1110111011 as the optimal scores. This would give a total score of $\boxed{38}$. □

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10th Annual Lexington Math Tournament - Guts Round - Part 9

Team Name: ______________________________

______________ 25. **[9]** Consider the equation $x^4-24x^3+210x^2+mx+n=0$. Given that the roots of this equation are nonnegative reals, find the maximum possible value of a root of this equation across all values of m and n.

Proposed by Andrew Zhao

Solution. $\boxed{9}$

Let the roots of the equation be a, b, c, and d. We are given that all roots are nonnegative, so Cauchy-Shwarz gives us $(a^2+b^2+c^2)(1^2+1^2+1^2) \geq (a+b+c)^2$. We can rewrite the left-hand side as $\big((a+b+c)^2 - 2(ab+bc+ca)\big)3$. By Vieta's, $a+b+c = 24-d$, and $ab+bc+ca = 210-d(a+b+c) = 210-d(24-d)$. Substituting these values into the inequality and simplifying, we get $d^2-12d+27 \leq 0$, or $3 \leq d \leq 9$. Hence, the maximum possible value of a root is $\boxed{9}$. Equality is achieved when $a=b=c=5$. □

26. **[9]** Let ω_1 and ω_2 be two circles with centers O_1 and O_2. The two circles intersect at A and B. ℓ is the circles' common external tangent that is closer to B, and it meets ω_1 at T_1 and ω_2 at T_2. Let C be the point on line AB not equal to A that is the same distance from ℓ as A is. Given that $O_1O_2 = 15$, $AT_1 = 5$ and $AT_2 = 12$, find $AC^2 + T_1T_2{}^2$.

Proposed by Zachary Perry

Solution. $\boxed{338}$

Let $AB \cap T_1T_2 = X$. By the Bisector Lemma, $XT_1 = XT_2$. Additionally, dropping the altitudes from X and A to ℓ gives $XA = XC$ by SAA congruence. Thus, AT_1CT_2 is a parallelogram, and it's well known that the sum of the squares of the diagonals is the sum of the squares of the sides. To prove this, try use LOC on the two supplementary angles. In this case, the wanted value is

$$2(5^2+12^2) = \boxed{338}.$$

□

27. **[9]** A list consists of all positive integers from 1 to 2020, inclusive, with each integer appearing exactly once. Define a move as the process of choosing four numbers from the current list and replacing them with the numbers $1, 2, 3, 4$. If the expected number of moves before the list contains exactly two 4's can be expressed as $\frac{a}{b}$ for relatively prime positive integers, evaluate $a+b$.

Proposed by Richard Chen and Taiki Aiba

Solution. $\boxed{1009}$

Call the number we want E. The only way we do not get two 4s on the first move is if one of the four elements we choose is the 4 in the original set. This happens with probability $\frac{\binom{2019}{3}}{\binom{2020}{4}} = \frac{1}{505}$ since we have $\binom{2019}{3}$ ways to choose the 3 elements other than 4.

Thus, we can set up the following state equation:

$$E = \frac{1}{505}(E+1) + \frac{504}{505} \cdot 1.$$

Solving, we get $E = \frac{505}{504}$, and our final answer is $505+504 = \boxed{1009}$. □

10th Annual Lexington Math Tournament - Guts Round - Part 10

Team Name: ______________________________

______________ 28. **[11]** 13 LHS Students attend the LHS Math Team tryouts. The students are numbered $1,2,..13$. Their scores are $s_1, s_2, ...s_{13}$, respectively. There are 5 problems on the tryout, each of which is given a weight, labelled $w_1, w_2, ...w_5$. Each score s_i is equal to the sums of the weights of all problems solved by student i. On the other hand, each weight w_j is assigned to be $\frac{1}{\sum s_i}$, where the sum is over all the scores of students who solved problem j. (If nobody solved a problem, the score doesn't matter). If the largest possible average score of the students can be expressed in the form $\frac{\sqrt{a}}{b}$, where a is square-free, find $a+b$.

Proposed by Jeff Lin

Solution. $\boxed{78}$

$\frac{sqrt65}{13}$ □

______________ 29. **[11]** Find the number of pairs of integers (a,b) with $0 \le a, b \le 2019$ where $ax \equiv b \pmod{2020}$ has exactly 2 integer solutions $0 \le x \le 2019$.

Proposed by Richard Chen

Solution. $\boxed{400}$

Note that the two solutions have to be x_1 and $x_1 + 1010$. If the answers are x_1 and $x_1 + c$ for some c, this indicates that $ac \equiv 0 \pmod{2020}$. If $c < 1010$, we have more than two integer solutions, since either $x_1 - c$ or $x_1 + 2c$ will be in the range. If $c > 1010$, we know that $a \cdot \gcd(c, 2020) = 2020$; in other words, $x_1 + \gcd(c, 2020)$ will also do the trick, showing again that we have more than two integer solutions. Because $1010a \equiv 0 \pmod{2020}$, and because we cannot have any other smaller divisor of 2020 to replace 1010, we know $\gcd(a, 2020) = 2$. From here, we find that $2a_1 \cdot x \equiv 2b_1 \pmod{2020} \implies a_1 \cdot x \equiv b_1 \pmod{1010}$(we can do this, which you can reason out yourself based on using the fact that mod is like a remainder) with $\gcd(a_1, 1010) = \gcd(b_1, 1010) = 1$. This equation actually only has one solution b_1 by the Chinese Remainder Theorem, so our final answer is a unique a determines a b, and we have $\phi(\frac{2020}{2}) = \phi(1010) = \frac{1}{2} \cdot \frac{4}{5} \cdot \frac{100}{101} = \boxed{400}$ such pairs (a,b). □

______________ 30. **[11]** $\triangle ABC$ has the property that $\angle ACB = 90°$. Let D and E be points on AB such that D is on ray BA, E is on segment AB, and $\angle DCA = \angle ACE$. Let the circumcircle of $\triangle CDE$ hit BC at $F \ne C$, and EF hit AC and DC at P and Q, respectively. If $EP = FQ$, then the ratio $\frac{EF}{PQ}$ can be written as $a + \sqrt{b}$ where a and b are positive integers. Find $a+b$.

Proposed by Kevin Zhao

Solution. $\boxed{11}$

Note that by letting $\angle ACE = \theta$ and $\angle ABC = \angle B$, then angle chasing yields that $\angle ACF = 90°$ and $\angle BCD = \angle BEF = 90° + \theta$. So, $\angle EFC = \angle EDC = 90° - \angle B - \theta$ and $\angle FEC = \angle B$ since the sides of a triangle sum up to $180°$. Similarly, we have that $\angle FQC = 2\theta + \angle B$ and $\angle CPE = 180° - \theta - \angle B$.

Now, we see that we apply Law of Sines to a bunch of triangles. This gets that $\frac{FQ}{FC} = \frac{\sin(90°-\theta)}{\sin(2\angle B+\theta)}$ and $\frac{FC}{FE} = \frac{\sin(\angle B)}{\sin(90°+\theta)}$ so

$$\frac{FQ}{FE} = \frac{FQ}{FC} \cdot \frac{FC}{FE} = \frac{\sin(90° - \theta)}{\sin(2\angle B + \theta)} \cdot \frac{\sin(\angle B)}{\sin(90° + \theta)} = \frac{\cos\theta}{\sin(2\angle B + \theta)} \cdot \frac{\sin(\angle B)}{\cos\theta} = \frac{\sin(\angle B)}{\sin(2\angle B + \theta)}.$$

In addition, $\frac{QP}{PE} = \frac{QC}{CE}$ by the Angle Bisector Theorem so since

$$\frac{QC}{CE} = \frac{\sin(\angle B)}{\sin(180° - 2\angle B - \theta)} = \frac{\sin(\angle B)}{\sin(2\angle B + \theta)},$$

then $\frac{FQ}{FE} = \frac{\sin(\angle B)}{\sin(2\angle B+\theta)} = \frac{QP}{PE}$ meaning $FQ \cdot PE = FE \cdot PQ$.

WLOG that $FQ = 1$. Because $FQ = PE$, then this equation simplifies to $FE \cdot PQ = 1$ meaning that $PQ(PQ+2) = 1$. Solving gets $PQ = 1+\sqrt{2}$ so

$$\frac{EF}{PQ} = \frac{\frac{1}{PQ}}{PQ} = PQ^{-2} = \frac{1}{3-2\sqrt{2}} = 3+\sqrt{8}$$

so our answer is $\boxed{11}$.

Remark. We note for any set of points F, Q, P, E and some C such that $\angle FCP = 90^\circ$ and $\angle QCP = \angle PCE$, then $(FP;QE) = -1$ is a harmonic pencil, meaning that $FQ \cdot PE = -QP \cdot EF$ if the sides are directed. If they are not directed, then $FQ \cdot PE = QP \cdot EF$. Knowing this fact, we can easily see that in this problem $(FP;QE) = -1$ and we can skip all the trigonometry. □

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10th Annual Lexington Math Tournament - Guts Round - Part 11

Team Name: ______________________________

______________ 31. **[13]** Let real angles $\theta_1, \theta_2, \theta_3, \theta_4$ satisfy

$$\sin\theta_1 + \sin\theta_2 + \sin\theta_3 + \sin\theta_4 = 0,$$
$$\cos\theta_1 + \cos\theta_2 + \cos\theta_3 + \cos\theta_4 = 0.$$

If the maximum possible value of the sum

$$\sum_{i<j} \sqrt{1 - \sin\theta_i \sin\theta_j - \cos\theta_i \cos\theta_j}$$

for $i, j \in \{1,2,3,4\}$ can be expressed as $a+b\sqrt{c}$, where c is square-free and a, b, c are positive integers, find $a+b+c$

Proposed by Alex Li

Solution. $\boxed{8}$

Consider the four unit vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$, where the x and y components of $\mathbf{v}_i$ are $\cos\theta_i$ and $\sin\theta_i$, respectively. The given equation can be rewritten as $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$. Geometrically, since all these vectors have equal length, the four vectors must form a rhombus when placed end-to-end. Thus, the four vectors must exist in two pairs pointing in opposite directions.

To find the desired sum, we can rewrite the summed expression as

$$\frac{\sqrt{(\sin\theta_i - \sin\theta_j)^2 + (\cos\theta_i - \cos\theta_j)^2}}{\sqrt{2}},$$

where the numerator denotes the distance between the arrowheads of vectors $\mathbf{v}_i$ and $\mathbf{v}_j$ when placed at the origin. The motivation for this expression comes from the $\sin\theta_i \sin\theta_j$ and $\cos\theta_i \cos\theta_j$ terms in the original expression. Thus our desired sum is the distances between any two distinct arrowheads when the vectors are placed at the origin, all divided by $\sqrt{2}$.

The distance between two pairs of vectors pointing in opposite directions is 2. Thus the summation is equal to $\frac{4+P}{\sqrt{2}}$, where P is the perimeter of the rectangle with vertices the arrowheads of the four vectors. We now wish to find the maximum value of $P = 2(x+y)$, where $x^2+y^2 = 4$ is the diagonal of the rectangle. It is clear this is maximized when $x = y$, and so $x = y = \sqrt{2}$ and $P = 4\sqrt{2}$. Thus the maximum value of our desired summation is $\frac{4+4\sqrt{2}}{\sqrt{2}} = 4+2\sqrt{2}$. The answer is $4+2+2 = \boxed{8}$. □

______________ 32. **[13]** In a lottery there are 14 balls, numbered from 1 to 14. Four of these balls are drawn at random. D'Angelo wins the lottery if he can split the four balls into two disjoint pairs, where the two balls in each pair have difference at least 5. The probability that D'Angelo wins the lottery can be expressed as $\frac{m}{n}$, with m, n relatively prime. Find $m+n$.

Proposed by Richard Chen

Solution. $\boxed{1506}$

We can translate the four balls into inequalities. Let the values on the balls be $a < b < c < d$. Then, we can denote $x_1 = a - 0, x_2 = b - a, x_3 = c - b, x_4 = d - c$. We have the following inequalities:

$$x_1, x_2, x_3, x_4 \geq 1$$

$$x_2 + x_3, x_3 + x_4 \geq 5$$

$$x_1 + x_2 + x_3 + x_4 \leq 14$$

Now, let $y_1 = x_1 - 1, y_2 = x_2 - 1, y_3 = x_3 - 1, y_4 = x_4 - 1$. We now have the new inequalities

$$y_1, y_2, y_3, y_4 \geq 0$$

$$y_2 + y_3, y_3 + y_4 \geq 3$$

$$y_1 + y_2 + y_3 + y_4 \leq 10$$

We can do casework on the value of $y_2 + y_3 + y_4$. For $y_2 + y_3 + y_4 = k$, we have $y_1 \leq 10 - k$, and thus we have $11 - k$ options for y_1 in this case. Furthermore, to satisfy the second set of inequalities, we can use complementary counting and PIE:

We have $\binom{k+2}{2}$ total ways for $y_2 + y_3 + y_4 = k$. Now, we count what doesn't work. $y_2 + y_3 = 0, 1, 2$ yield $1, 2, 3$ invalid solutions respectively, as does $y_3 + y_4 = 0, 1, 2$. So, we have a total of $\binom{k+2}{2} - 12$ working triplets of $y_2 + y_3 + y_4$.

However, we need to be careful about $k = 3, 4$, since $y_2 + y_3 = 0, 1, 2$ and $y_3 + y_4 = 0, 1, 2$ will have some overlap. No worries! These cases are quick to get by themselves. $k = 3$ yields $(y_2, y_3, y_4) = (0, 3, 0)$ and $k = 4$ yields $(y_2, y_3, y_4) = (0, 4, 0), (1, 3, 0), (0, 3, 1), (1, 2, 1)$.

So, our winning ways are

$$(11-3)\cdot 1 + (11-4)\cdot 4 + \sum_{i=5}^{10}(11-i)\left[\binom{i+2}{2} - 12\right] = 505$$

with a probability of $\frac{505}{1001}$, so $505 + 1001 = \boxed{1506}$. □

33. **[13]** Let ω_1 and ω_2 be two circles that intersect at two points: A and B. Let C and E be on ω_1, and D and F be on ω_2 such that CD and EF meet at B and the three lines CE, DF, and AB concur at a point P that is closer to B than A. Let Ω denote the circumcircle of $\triangle DEF$. Now, let the line through A perpendicular to AB hit EB at G, GD hit Ω at J, and DA hit Ω again at I. A point Q on IE satisfies that $CQ = JQ$. If $QJ = 36$, $EI = 21$, and $CI = 16$, then the radius of Ω can be written as $\frac{a\sqrt{b}}{c}$ where a, b, and c are positive integers, b is not divisible by the square of a prime, and $\gcd(a, c) = 1$. Find $a + b + c$.

Proposed by Kevin Zhao

Solution. $\boxed{1290}$

First, we note that $PF \cdot PD = PA \cdot PB = PE \cdot PC$ meaning that by the converse of the Power of a Point, C lies on Ω. Now, we see that

$$\angle DAB = \angle DFE = \angle DCE = \angle BCE = \angle BAE$$

which shows that $\angle DAB = \angle BAE$. Letting AD and BF hit at K, we see that since we have that $\angle BAE = \angle DAB = \angle KAB$ and $\angle GAB = 90°$, then $-1 = (EF; KG)$ and taking perspectivity with respect to D and Ω, we get that $-1 = (EK; BG) = (EI; CJ)$. As a result of that, we see that the tangents with respect to Ω passing through J and C hit on IE.

This shows us that since $QJ = QC = 36$, then $QE \cdot QI = 36^2 = 1296$ and $QE(QE + EI) = QE(QE + 21) = 1296$. Solving gets us that $QE = 27$ and so because $\triangle QCE \sim \triangle QIC$, then $EC = CI \cdot \frac{QE}{QC} = 16 \cdot \frac{27}{36} = 12$.

We now see that the area of $\triangle CIE$ is, by Heron's, $\sqrt{s(s-a)(s-b)(s-c)}$ where $s = 0.5(16+21+12) = 24.5$ and $(a,b,c) = (16,21,12)$ in any order. So, our area is just

$$\sqrt{24.5(8.5)(3.5)(12.5)} = \frac{\sqrt{49 \cdot 17 \cdot 7 \cdot 25}}{2} = \frac{35\sqrt{119}}{4}$$

and now note that Ω is also the circumcircle of $\triangle CIE$. So, our circumradius is, where A is the area of $\triangle CIE$, written as

$$\frac{abc}{4A} = \frac{16 \cdot 21 \cdot 12}{4 \cdot \frac{35\sqrt{119}}{4}} = \frac{16 \cdot 3 \cdot 12}{5\sqrt{119}} = \frac{576\sqrt{119}}{595}$$

and summing up our numbers, our answer is $576 + 119 + 595 = \boxed{1290}$. $\square$

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10th Annual Lexington Math Tournament - Guts Round - Part 12

Team Name: ______________________________

______________ 34. **[15]** Your answer to this problem will be an integer between 0 and 100, inclusive. From all the teams who submitted an answer to this problem, let the average answer be A. Estimate the value of $\lfloor \frac{2}{3} A \rfloor$. If your estimate is E and the answer is A, your score for this problem will be

$$\max(0, \lfloor 15 - 2 \cdot |A - E| \rfloor).$$

Proposed by Andrew Zhao

Solution. $\boxed{}$

□

______________ 35. **[15]** Estimate the number of ordered pairs (p, q) of positive integers at most 2020 such that the cubic equation $x^3 - px - q = 0$ has three distinct real roots. If your estimate is E and the answer is A, your score for this problem will be

$$\left\lfloor 15 \min\left(\frac{A}{E}, \frac{E}{A}\right)\right\rfloor.$$

Proposed by Alex Li

Solution. $\boxed{365127}$

In order for the cubic to have three distinct real roots, the local minimum must sit below the x-axis. At the local minimum, the derivative is zero:

$$\frac{d}{dx}(x^3 - px - q) = 3x^2 - p = 0$$

Thus $x = \sqrt{\frac{p}{3}}$ (the negative root corresponds to the local maximum). Plugging this value of x into the cubic to obtain the y-value gives

$$y = \frac{p}{3}\sqrt{\frac{p}{3}} - p\sqrt{\frac{p}{3}} - q = -\frac{2p}{3}\sqrt{\frac{p}{3}} - q < 0.$$

Squaring the final inequality gives

$$4p^3 < 27q^2 \Rightarrow p < \frac{3}{2^{2/3}} q^{2/3}.$$

Thus the number of possible values of p that satisfy the given condition is equal to

$$\left\lfloor \frac{3}{2^{2/3}} q^{2/3} \right\rfloor$$

and the total number of ordered pairs is given by the sum

$$\sum_{q=1}^{2020} \left\lfloor \frac{3}{2^{2/3}} q^{2/3} \right\rfloor.$$

Ignoring the floor function allows us to approximate this summation as a right Riemann sum; that is,

$$\sum_{q=1}^{2020} \frac{3}{2^{2/3}} q^{2/3} \approx \frac{3}{2^{2/3}} \int_0^{2020} q^{2/3}\, dq.$$

Evaluating and approximating the integral gives

$$\begin{aligned}\frac{3}{2^{2/3}}\int_0^{2020} q^{2/3}\,dq &= \frac{3}{2^{2/3}}\cdot\frac{3}{5}\cdot(2020)^{5/3}\\ &\approx \frac{3^2\cdot 2^{5/3}\cdot 1000^{5/3}}{2^{2/3}\cdot 5}\\ &= \frac{18}{5}\cdot 100000\\ &= 360000.\end{aligned}$$

Given the real answer is 365127, the approximation is very accurate. □

36. **[15]** Estimate the product of all the nonzero digits in the decimal expansion of 2020!. If your estimate is E and the answer is A, your score for this problem will be

$$\max\left(0, \left\lfloor 15 - 0.02\cdot\left|\log_{10}\left(\frac{A}{E}\right)\right|\right\rfloor\right).$$

Proposed by Alex Li

Solution. $\boxed{2.77842640466\cdot 10^{2959}}$

This problem can be estimated accurately by assuming aside from terminal zeros, the digits 0 through 9 are evenly distributed throughout the decimal expansion. Thus, in order to generate an accurate estimate, we must compute the number of digits in the expansion of 2020!

The number of digits is equivalent to

$$\lfloor\log_{10}(2020!)\rfloor \approx \sum_{i=1}^{2020}\log_{10}(i).$$

The summation can be interpreted as a right Riemann sum, and can be approximated with its respective integral:

$$\sum_{i=1}^{2020}\log_{10}(i) \approx \int_0^{2020}\log_{10} x\,dx = \frac{2020(\ln 2020 - 1)}{\ln 10}.$$

Using $e^3 \approx 20$ and $\ln 2 \approx 0.7$, which are common approximations, we can estimate

$$\ln 10 \approx \ln 20 - \ln 2 \approx 2.3$$

and

$$\ln 2020 \approx 11\ln 2 \approx 7.7$$

Thus the number of digits is approximately $2020\cdot 3 \approx 6000$. The number of terminal zeros can be calculated as

$$\sum_{i=1}^{\infty}\left\lfloor\frac{2020}{5^i}\right\rfloor = 404+80+16+3 = 503.$$

Thus the number of non-terminal zeros is approximately $6000 - 500 = 5500$. Among these digits, each digit 0 through 9 appears approximately 550 times. Thus the product of the nonzero digits is approximately $(9!)^{550}$.

The actual answer was calculated by computer and was found to be $2.78\cdot 10^{2959}$, the log of which is 2959.44. The log of the estimate is $\log_{10}(9!)^{550} = 550\log_{10}(9!) \approx 3058$, which is very close to the actual answer, showing that our approximation is valid. □