Division B Team Round

Lexington High School

December 5th, 2020

1. **[10]** Four Ls are equivalent to three Ms. Nine Ms are equivalent to fourteen Ts. Seven Ts are equivalent to two Ws. If Kevin has thirty-six Ls, how many Ws would that be equivalent to?

   Proposed by Kevin Zhao

   Solution. $\boxed{12}$

   36 Ls are equivalent to 27 Ws which is also 42 Ts and thus $\boxed{12}$ Ws. □

2. **[10]** The area of a square is 144. An equilateral triangle has the same perimeter as the square. The area of a regular hexagon is 6 times the area of the equilateral triangle. What is the perimeter of the hexagon?

   Proposed by Euhan Kim

   Solution. $\boxed{96}$

   The side length of the square is 12, so its perimeter is 48. Therefore, the side length of the triangle is 16. Since the hexagon's area is 6 times larger, we can simply combine 6 of the equilateral triangles together to create the hexagon. The hexagon's side length is the same as the triangle's. Thus, the answer is $6 \cdot 16 = \boxed{96}$. □

3. **[10]** Find the number of ways to arrange the letters in $LEXINGTON$ such that the string LEX does not appear.

   Proposed by Ada Tsui

   Solution. $\boxed{178920}$

   Disregarding the restriction that the string LEX must not appear, the number of ways to arrange the letters in $LEXINGTON$ is $\frac{9!}{2!} = 181440$ (note that N repeats twice in $LEXINGTON$).

   If the string LEX does indeed appear, then the number of ways to arrange the letters in $LEXINGTON$ is $\frac{7!}{2!} = 360$ (note that we can treat the string LEX as a letter and once again, N repeats twice in $LEXINGTON$).

   Thus, the number of ways to arrange the letters in $LEXINGTON$ such that the string LEX does not appear is $\boxed{178920}$. □

4. **[10]** Find the greatest prime factor of $20! + 20! + 21!$.

   Proposed by Ada Tsui

   Solution. $\boxed{23}$

   Factor out $20!$ from $20! + 20! + 21!$ to get $20!(1+1+21) = 20! \cdot 23$. The greatest prime factor in the first factor is 19, and the greatest prime factor in the second factor is 23, so the greatest prime factor of the expression is $\boxed{23}$. □

5. **[15]** Given the following system of equations

$$\begin{aligned} a_1 + a_2 + a_3 &= 1 \\ a_2 + a_3 + a_4 &= 2 \\ a_3 + a_4 + a_5 &= 3 \\ &\vdots \\ a_{12} + a_{13} + a_{14} &= 12 \\ a_{13} + a_{14} + a_1 &= 13 \\ a_{14} + a_1 + a_2 &= 14, \end{aligned}$$

find the value of a_{14}.

Proposed by Audrey Chun

Solution. $\boxed{9}$

By adding the equations, we get $3(a_1 + a_2 + \ldots + a_{13} + a_{14}) = 1 + 2 + 3 \ldots + 14$, so $a_1 + a_2 + \ldots + a_{13} + a_{14} = 35$. By being clever, you'd find that adding the following 5 equations

$$\begin{aligned} a_{14} + a_1 + a_2 &= 14 \\ a_3 + a_4 + a_5 &= 3 \\ a_6 + a_7 + a_8 &= 6 \\ a_9 + a_{10} + a_{11} &= 9 \\ a_{12} + a_{13} + a_{14} &= 12 \end{aligned}$$

will give you $a_{14} + a_1 + a_2 + \ldots + a_{13} + a_{14} = 44$. Therefore, $\boxed{a_{14} = 9}$. □

6. **[15]** 1001 marbles are drawn at random and without replacement from a jar of 2020 red marbles and n blue marbles. Find the smallest positive integer n such that the probability that there are more blue marbles chosen than red marbles is strictly greater than $\frac{1}{2}$.

Proposed by Taiki Aiba

Solution. $\boxed{2021}$

Note that if $n = 2020$, then the probability of choosing more blue than red is equal to the probability of choosing more red than blue. It is easy to see that when $n > 2020$, the probability of choosing more blue than red is greater than the probability of choosing more red than blue, so the answer is $\boxed{2021}$. □

7. **[15]** Zachary tries to simplify the fraction $\frac{2020}{5050}$ by dividing the numerator and denominator by the same integer to get the fraction $\frac{m}{n}$, where m and n are both positive integers. Find the sum of the (not necessarily distinct) prime factors of the sum of all the possible values of $m + n$.

Proposed by Ada Tsui

Solution. $\boxed{37}$

The new fraction $\frac{m}{n}$ must be in the form of $\frac{2d}{5d}$ where d is a positive integer that divides 1010.

Thus, the sum of all possible values of $m + n$ is seven times the sum of the divisors of 1010.

From here, all we need to do is perform the calculations. The sum of the divisors of 1010 is $(2^0 + 2^1)(5^0 + 5^1)(101^0 + 101^1) = 3 \cdot 6 \cdot 102 = 2^2 \cdot 3^3 \cdot 17$. Multiplying that by 7 gives $2^2 \cdot 3^3 \cdot 7 \cdot 17$.

Of course, we don't want the sum of the possible values of $m + n$; we want the sum of the prime factors of the possible values of $m + n$, which is $2 + 2 + 3 + 3 + 3 + 7 + 17 = \boxed{37}$. □

8. **[15]** In rectangle $ABCD$, $AB = 3$ and $BC = 4$. If the feet of the perpendiculars from B and D to AC are X and Y, the length of XY can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Proposed by Ada Tsui

Solution. $\boxed{12}$

Find the length of AC to be 5 since ABC is a right triangle of legs 3 and 4.

Triangles ABX and ACB are similar because they share the angle A and have right angles at X and B, respectively. Thus, triangle ABX's side lengths also have a ratio of $3:4:5$. Since we already know that $AB = 3$, $AX = 3 \cdot \frac{3}{5} = \frac{9}{5}$.

Triangles ABX and CDY are congruent because BX, DY are both altitudes of the congruent triangles ABC and ADC (due to rectangle $ABCD$, $AB = CD$ and $AD = BC$, and they share the side AC), $AB = CD$, and they have right angles at X and Y. By CPCTC, $CY = AX$, or $\frac{9}{5}$.

We are looking for XY, which is $AC - AX - CY$. We have all of these values! So $XY = 5 - \frac{9}{5} - \frac{9}{5} = \frac{7}{5}$.

Thus, $m = 7, n = 5$, and $m + n = 7 + 5 = \boxed{12}$. □

9. **[20]** Ben writes the string

$$\underbrace{111\ldots11}_{\text{2020 digits}}$$

on a blank piece of paper. Next, in between every two consecutive digits, he inserts either a plus sign (+) or a multiplication sign (×). He then computes the expression using standard order of operations. Find the number of possible distinct values that Ben could have as a result.

Proposed by Taiki Aiba

Solution. $\boxed{2020}$

Note that at the end, we have anything between 1 one and 2020 ones, inclusive. Thus, our answer is $\boxed{2020}$. □

10. **[20]** In a certain Zoom meeting, there are 4 students. How many ways are there to split them into any number of distinguishable breakout rooms, each with at least 1 student?

Proposed by Jeff Lin

Solution. $\boxed{75}$

We can do casework on the number of breakout rooms. For one room, the answer is clearly 1. For 2, we have 2^4 ways, but 2 of those 16 ways has an empty room, so there are 14 ways. For 3 rooms, we get $3^4 = 81$ ways, but $3 * 2^4 - 3 = 45$ of those ways has an empty room (PIE). This leaves 36 ways. Then, for 4 rooms, the split must be $1-1-1-1$, so there are $4 * 3 * 2 * 1 = 24$ ways. Adding this all up, we get $1 + 14 + 36 + 24 = \boxed{75}$ □

11. **[20]** $\triangle ABC$ is an isosceles triangle with $AB = AC$. Let M be the midpoint of BC and E be the point on AC such that $AE : CE = 5 : 3$. Let X be the intersection of BE and AM. Given that the area of $\triangle CMX$ is 15, find the area of $\triangle ABC$.

Proposed by Euhan Kim

Solution. $\boxed{130}$

Since $\triangle ABC$ is an isosceles triangle and X lies on AM, the area of $\triangle CMX$ and the area of $\triangle BMX$ are equal. Therefore, the area of BXC is 30. Using mass points, we can find that the ratio of $AM : MX = 13 : 3$. Since $\triangle ABC$ and $\triangle BXC$ share the same base, we use the ratio of the heights for both triangles to find the area of $\triangle ABC$.

$$\triangle ABC = \triangle BXC \cdot \frac{13}{3} = 30 \cdot \frac{13}{3} = \boxed{130}$$

□

12. **[20]** Find the sum of all positive integers a such that there exists an integer n that satisfies the equation:

$$a! \cdot 2^{\lfloor \sqrt{a} \rfloor} = n!.$$

Proposed by Ivy Zheng

Solution. $\boxed{32}$

Note a must be one less than $2^{\lfloor \sqrt{a} \rfloor} = n$, and $n = a + 1$. This means $a = 2^{\lfloor \sqrt{a} \rfloor} - 1$. Listing first few powers of 2, we see that only $a = 1, 31$ work, so we have $1 + 31 = \boxed{32}$. □

13. **[25]** Compute the number of ways there are to completely fill a 3×15 rectangle with non-overlapping 1×3 rectangles.

Proposed by Alex Li

Solution. $\boxed{189}$

Let a_n be the number of ways to fill a $3 \times n$ rectangle with 3×1 rectangles. Consider an arbitrary $3 \times n$ rectangle. Look at the rightmost column of the rectangle. If it is a vertical rectangle, then the number of ways to tile it is a_{n-1}. If it is the ends of three horizontal rectangles, the number of ways to tile the rest is a_{n-3}. This implies the recursion

$$a_n = a_{n-1} + a_{n-3}, \forall a \geq 4.$$

Using $a_1 = 1, a_2 = 1$, and $a_3 = 2$, we find $a_{15} = \boxed{189}$ after a lot of computation. □

14. **[25]** Let $\triangle ABC$ with $AB = AC$ and $BC = 14$ be inscribed in a circle ω. Let D be the point on ray BC such that $CD = 6$. Let the intersection of AD and ω be E. Given that $AE = 7$, find AC^2.

    Proposed by Ephram Chun and Euhan Kim

    Solution. $\boxed{105}$

    We use Power of Point with point D. Then, we get that $DE\cdot(7+DE) = 6\cdot(6+14)$. So, $DE = 8$ and $AD = 15$.
    Now, we drop the altitude of $\triangle ABC$ from A to BC and call that point M. Since $\triangle ABC$ is an isosceles triangle, M is the midpoint of BC. We simply use Pythagorean Theorem on $\triangle AMD$:

$$AD^2 = MD^2 + AM^2$$

$$15^2 = (6+7)^2 + AM^2$$

$$225 = 169 + AM^2$$

$AM = \sqrt{56}$. Then, we use Pythagorean Theorem on $\triangle AMC$.

$$AC^2 = AM^2 + MC^2$$

$$AC^2 = (\sqrt{56})^2 + 7^2 = 56 + 49 = \boxed{105}.$$

□

15. **[25]** Let S denote the sum of all rational numbers of the form $\frac{a}{b}$, where a and b are relatively prime positive divisors of 1300. If S can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers, then find $m+n$.

    Proposed by Ephram Chun

    Solution. $\boxed{2819}$

    The sum is equivalent to $\sum\limits_{i|1300^2} \frac{i}{1300}$ when i is all the factors of 1300^2 because when $\frac{i}{1300^2}$ is reduced it will be a reduced fraction and both numbers will be relatively prime to each other and be factors of 1300. Since it a can be greater or less than b we need the factors of 1300^2. Thus our answer is

    $\frac{(1+13)(1+2+4)(1+5+25)}{1300} = \frac{14*7*31}{1300} = \frac{7*7*31}{1300} = \boxed{\frac{1519}{1300}}$ We are looking for $m+n$, which is $1519+1300 = \boxed{2819}$. □

16. **[25]** Let f be a function $\mathbb{R}\to\mathbb{R}$ that satisfies the following equation:

$$f(x)^2 + f(y)^2 = f(x^2+y^2) + f(0)$$

If there are n possibilities for the function, find the sum of all values of $n\cdot f(12)$.

Proposed by Zachary Perry

Solution. $\boxed{39}$

First, we subtract $2f(0)$ from both sides. Let $g(x) = f(x) - f(0)$. Substituting in gives

$$g(x)^2 + g(y)^2 = g(x^2+y^2).$$

Let $P(x,y)$ be this assertion. Then $P(0,0)$ gives either $g(0) = 0$ or $g(0) = 1$. We will use casework.

Assume $g(0) = 0$ for now. Then, $P(x,0)$ gives $g(x)^2 = g(x^2)$. We substitute both $f(x)^2 = f(x^2)$ and $f(y)^2 = f(y^2)$ in, and then we have Cauchy's functional equation. As well we know that $g(x)$ is bounded below on positives, because $g(x^2) = g(x)^2 \geq 0$. So, $g(x)$ is linear.

If $g(0) = 1$, then $P(x,0)$ gives the same thing as it did in the first case, but with an extra $+1$. We use the same trick at the beginning; letting $h(x) = g(x) - 1$. From there, we do exactly what we did in the first case, and we conclude that $h(x)$ is linear again, which implies $g(x)$ is too.

Because $g(x)$ is linear, $f(x)$ is too. Now all that remains is to test which linear functions work. Letting $f(x) = ax+b$ and substituting into the given shows that the only working functions are $f(x) = x$, $f(x) = 0$, and $f(x) = 1$ Thus the requested answer is $3\cdot(0+1+12) = \boxed{39}$. □

17. **[30]** Circle ω has radius 10 with center O. Let P be a point such that $PO = 6$. Let the midpoints of all chords of ω through P bound a region of area R. Find the value of $\lfloor 10R \rfloor$.

Proposed by Andrew Zhao

Solution. $\boxed{282}$

Let the midpoint of any given chord be M. We have $\angle PMO = 90°$, so the circumcircle of $\triangle PMO$ is a circle with diameter PO. Hence, all points M lie on this circle, and the area of this region is $\pi\left(\frac{PO}{2}\right)^2 = 9\pi$, so the answer is $\boxed{282}$. □

18. **[30]** Define a sequence $\{a_n\}_{n\geq 1}$ recursively by $a_1 = 1$, $a_2 = 2$, and for all integers $n \geq 2$, $a_{n+1} = (n+1)^{a_n}$. Determine the number of integers k between 2 and 2020, inclusive, such that $k+1$ divides $a_k - 1$.

Proposed by Taiki Aiba

Solution. $\boxed{1009}$

The condition should hold if and only if k is odd. Note that $a_k = k^{a_{k-1}}$. It should be clear that a_k and k have the same parity, and $k \equiv -1 \pmod{k+1}$. Combining these two facts yields that $a_k \equiv (-1)^{k-1 \pmod 2}$. If k is even, then $a_k \equiv -1 \pmod{k+1}$, so $a_k - 1 \equiv -2 \pmod{k+1}$, and if k is odd, then $a_k \equiv 1 \pmod{k+1}$, so $a_k - 1 \equiv 0 \pmod{k+1}$, which is what we want. So, k must be odd for the conditions to hold. We count all of the odd numbers from 3 to 2019, inclusive, for a total of $\boxed{1009}$ numbers that work. □

19. **[30]** Ada is taking a math test from 12:00 to 1:30, but her brother, Samuel, will be disruptive for two ten-minute periods during the test. If the probability that her brother is not disruptive while she is solving the challenge problem from 12:45 to 1:00 can be expressed as $\frac{m}{n}$, find $m+n$.

Proposed by Ada Tsui

Solution. $\boxed{281}$

We use geometric probability based on when Samuel starts to be disruptive.

The viable area is a 80 minute by 80 minute square with the diagonal strip in the center omitted so that the two 10 minute periods do not overlap with an area of 70^2.

The favorable area is the viable area without the area where the x or the y coordinates are between 35 and 60, because that is when Ada is doing the challenge problem.

Calculating the probability, we find $\frac{2\cdot(20\cdot35)+(35-10)^2+(20-10)^2}{70^2} = \frac{85}{196}$.

Thus, $m = 85, n = 196$, and $m+n = 85+196 = \boxed{281}$. □

20. **[30]** Two sequences of nonzero reals $a_1, a_2, a_3, \dots$ and $b_2, b_3, \dots$ are such that $b_n = \prod_{i=1}^{n} a_i$ and $a_n = \frac{b_n^2}{3b_n-3}$ for all integers $n > 1$. Given that $a_1 = \frac{1}{2}$, find $|b_{60}|$.

Proposed by Andrew Zhao

Solution. $\boxed{3}$

$a_n = \frac{b_n}{b_{n-1}} \implies \frac{b_n}{b_{n-1}} = \frac{{b_n}^2}{3b_n-3}$. From here we can divide by b_n because it's never 0, and get $\frac{1}{b_{n-1}} = \frac{b_n}{3b_n-3}$. Rearranging gives $b_n = \frac{3}{3-b_{n-1}}$, and then notice it has period 6. This means $b_{60} = b_6$, and calculating this gives us $b_6 = -3$, so our answer is $\boxed{3}$. □

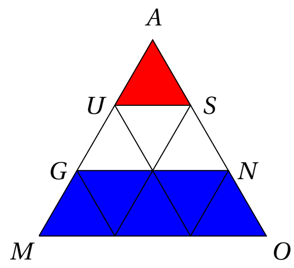
Among Us

21. **[15]** Let $\triangle AMO$ be an equilateral triangle. Let U and G lie on side AM, and let S and N lie on side AO such that $AU = UG = GM$ and $AS = SN = NO$. Find the value of $\frac{[MONG]}{[USA]}$.

Proposed by Zachary Perry

Solution. $\boxed{5}$

Split it into nine congruent equilateral triangles. Note that $MONG$ contains 5 of these, and USA is 1 of the triangles, so the answer is just $\boxed{5}$.

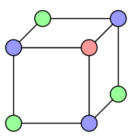


□

22. **[20]** A cube has one of its vertices and all edges connected to that vertex deleted. How many ways can the letters from the word "AMONGUS" be placed on the remaining vertices of the cube so that one can walk along the edges to spell out "AMONGUS"? Note that each vertex will have at most 1 letter, and one vertex is deleted and not included in the walk.

Proposed by Samuel Charney

Solution. $\boxed{24}$



We use casework on whether the A is at a red, green, or blue point.

If A is at the red point, then we have three symmetric choices for where to put the M; choose any one of these blue points. From here, putting O at any one of the two adjacent green points gives one way, so we have 6 paths from the red point.

If A is one of the green points, we have two symmetric choices for the M; choose any one of these blue points. If we then go to the adjacent green point for O, we have two ways to finish(N,G,U,S at B,R,B,G or B,G,B,R). If we go to the adjacent red point instead for O, we have one way to finish. So, each of these points has $2 \cdot (2+1) = 6$ paths. We have a total of 18 paths here because we have 3 green points we could've started at.

If A is one of the blue points, we see that we have no paths.

Thus, we have $6+18 = \boxed{24}$ total ways to place our letters. □

23. **[20]** The LHS Math Team wants to play Among Us. There are so many people who want to play that they are going to form several games. Each game has at most 10 people. People are *happy* if they are in a game that has at least 8 people in it. What is the largest possible number of people who would like to play Among Us such that it is impossible to make everyone *happy*?

Proposed by Samuel Charney

Solution. $\boxed{31}$

This is equivalent to finding the largest number that cannot be formed by adding multiples of 8, 9, and 10. Considering these sums modulo 8, it is clear that the hardest remainder to achieve mod 8 is 7, with the smallest achievable being 39. Therefore, the largest number of people such that not everyone is happy is $39-8 = \boxed{31}$. □

24. **[25]** In a game of Among Us, there are 10 players and 12 colors. Each player has a "default" color that they will automatically get if nobody else has that color. Otherwise, they get a random color that is not selected. If 10 random players with random default colors join a game one by one, the expected number of players to get their default color can be expressed as $\frac{m}{n}$. Compute $m+n$. Note that the default colors are not necessarily distinct.

Proposed by Jeff Lin

Solution. $\boxed{29}$

We realize that the first person is guaranteed to get their default color, the second person has a $\frac{11}{12}$ chance, the third person has a $\frac{10}{12}$ chance, and so on. Thus, the answer is just $1+\frac{11}{12}+\frac{10}{12}+...\frac{3}{12}=\frac{75}{12}=\frac{25}{4}$. $25+4=\boxed{29}$. □

Radishes

25. **[15]** Emmy goes to buy radishes at the market. Radishes are sold in bundles of 3 for \$5 and bundles of 5 for \$7. What is the least number of dollars Emmy needs to buy exactly 100 radishes?

Proposed by Alex Li

Solution. $\boxed{140}$

The value of the bundle of three radishes, 1.67 dollars per radish, is less than the the value of the bundle of five radishes, 1.40 dollars per radish.

Thus, Emmy wants to buy as many bundles of five radishes as she can. Luckily for us, this works out to 20 bundles of five radishes for the 100 radishes she needs to buy.

That price is just $20 \cdot 7 = \boxed{140}$ dollars. □

26. **[20]** Aidan owns a plot of land that is in the shape of a triangle with side lengths 5,10, and $5\sqrt{3}$ feet. Aidan wants to plant radishes such that there are no two radishes that are less than 1 foot apart. Determine the maximum number of radishes Aidan can plant.

Proposed by Ephram Chun

Solution. $\boxed{36}$

Note that the triangle is a 30-60-90 triangle. The most efficient way to place the radishes is if they are placed in a triangular fashion. If we place 6 radishes along the side of length 5, and 11 radishes along the side of length 10, we can continue to place radishes such that equilateral triangles with side length 1 are created. Adding this up, we get $\boxed{36}$ radishes. □

27. **[25]** Alex and Kevin are radish watching. The probability that they will see a radish within the next hour is $\frac{1}{17}$. If the probability that they will see a radish within the next 15 minutes is p, determine $\lfloor 1000p \rfloor$. Assume that the probability of seeing a radish at any given moment is uniform for the entire hour.

Proposed by Ephram Chun

Solution. $\boxed{492}$

Let the probability of seeing a radish in the next 15 minutes be p.

$$p^4 = \frac{1}{17}$$

$$p \approx 0.49247$$

$$\lfloor 1000p \rfloor = \lfloor 1000 * 0.49247 \rfloor = \boxed{492}.$$

□

COVID

28. **[15]** There are 2500 people in Lexington High School, who all start out healthy. After 1 day, 1 person becomes infected with coronavirus. Each subsequent day, there are twice as many newly infected people as on the previous day. How many days will it be until over half the school is infected?

Proposed by Sam Charney

Solution. $\boxed{11}$

On Day 1, 1 person is infected. On Day 2, 2 more people are infected, for a total of 3. On Day n, 2^{n-1} people are infected for a total of $2^n - 1$. The first n such that $2^n - 1 > 2500$ is $n = \boxed{11}$. □

29. **[20]** Alicia bought some number of disposable masks, of which she uses one per day. After she uses each of her masks, she throws out half of them (rounding up if necessary) and reuses each of the remaining masks, repeating this process until she runs out of masks. If her masks lasted her 222 days, how many masks did she start out with?

    Proposed by Jeff Lin

    Solution. $\boxed{113}$

    It is reasonably easy to check whether each amount of masks work, so we just check every possible number and see that $\boxed{113}$ works, so that is the answer. □

30. **[25]** Arthur has a regular 11-gon. He labels the vertices with the letters in $CORONAVIRUS$ in consecutive order. Every non-ordered set of 3 letters that forms an isosceles triangle is a member of a set S, i.e. {C, O, R} is in S. How many elements are in S?

    Proposed by Samuel Charney

    Solution. $\boxed{53}$

    We need to remove the number of double counted sets from the total number of sets. The total number of isosceles triangles is $11 \cdot 5 = 55$ because you pick the top vertex out of 11 vertices and then have 5 choices for the height. A set is double counted if you can use the other O and or the other R to still get an isosceles triangle. If two isosceles triangles share a side in common and have each O as their third vertex, the four points must form a parallelogram. However this is impossible as the total number of points is not a multiple of 4. The same is true for switching with R is included. Now for switching the O and the R. The isosceles triangles with R_1, not R_2, and only one of the Os are CO_1R_1, R_1O_2N, and O_1R_1I. CO_2R_2 and R_2O_1N are both isosceles so we remove them from the count, while O_2R_2I is not. Therefore, we only have 2 double counted sets, for a final answer of $\boxed{53}$. □