Division A Team Round

Lexington High School

December 5th, 2020

1. **[10]** Ben writes the string

$$\underbrace{111\dots11}_{2020\text{ digits}}$$

on a blank piece of paper. Next, in between every two consecutive digits, he inserts either a plus sign (+) or a multiplication sign (×). He then computes the expression using standard order of operations. Find the number of possible distinct values that Ben could have as a result.

Proposed by Taiki Aiba

Solution. $\boxed{2020}$

Note that at the end, we have anything between 1 one and 2020 ones, inclusive. Thus, our answer is $\boxed{2020}$. □

2. **[10]** 1001 marbles are drawn at random and without replacement from a jar of 2020 red marbles and n blue marbles. Find the smallest positive integer n such that the probability that there are more blue marbles chosen than red marbles is strictly greater than $\frac{1}{2}$.

Proposed by Taiki Aiba

Solution. $\boxed{2021}$

Note that if $n = 2020$, then the probability of choosing more blue than red is equal to the probability of choosing more red than blue. It is easy to see that when $n > 2020$, the probability of choosing more blue than red is greater than the probability of choosing more red than blue, so the answer is $\boxed{2021}$. □

3. **[10]** Find the value of $\lfloor \frac{1}{6} \rfloor + \lfloor \frac{4}{6} \rfloor + \lfloor \frac{9}{6} \rfloor + \cdots + \lfloor \frac{1296}{6} \rfloor$.

Proposed by Zachary Perry

Solution. $\boxed{2688}$

Notice that the fractional parts of each term form a pattern. This is because if a is an integer, then

$$\left\{\frac{a^2}{6}\right\} = \left\{\frac{a^2}{6} + 2a + 6\right\} = \left\{\frac{(a+6)^2}{6}\right\}.$$

Thus, they cycle in this pattern: $\frac{1}{6}, \frac{4}{6}, \frac{3}{6}, \frac{4}{6}, \frac{1}{6}, 0 \dots$. Therefore, after every 6 terms, there is $\frac{13}{6}$ "excess" not counted. We can write the wanted expression as the sum minus the fractional part. Using the sum of squares formula, we have

$$\frac{\frac{36\cdot 37\cdot 73}{6}}{6} - \frac{13}{6}\cdot 6 = 37\cdot 73 - 13 = \boxed{2688}.$$

□

4. **[10]** Let $\triangle ABC$ with $AB = AC$ and $BC = 14$ be inscribed in a circle ω. Let D be the point on ray BC such that $CD = 6$. Let the intersection of AD and ω be E. Given that $AE = 7$, find AC^2.

Proposed by Ephram Chun and Euhan Kim

Solution. $\boxed{105}$

We use Power of Point with point D. Then, we get that $DE\cdot(7+DE)=6\cdot(6+14)$. So, $DE=8$ and $AD=15$.
Now, we drop the altitude of $\triangle ABC$ from A to BC and call that point M. Since $\triangle ABC$ is an isosceles triangle, M is the midpoint of BC. We simply use Pythagorean Theorem on $\triangle AMD$:

$$AD^2=MD^2+AM^2$$

$$15^2=(6+7)^2+AM^2$$

$$225=169+AM^2$$

$AM=\sqrt{56}$. Then, we use Pythagorean Theorem on $\triangle AMC$.

$$AC^2=AM^2+MC^2$$

$$AC^2=(\sqrt{56})^2+7^2=56+49=\boxed{105}.$$

□

5. **[15]** Ada is taking a math test from 12:00 to 1:30, but her brother, Samuel, will be disruptive for two ten-minute periods during the test. If the probability that her brother is not disruptive while she is solving the challenge problem from 12:45 to 1:00 can be expressed as $\frac{m}{n}$, find $m+n$.

   Proposed by Ada Tsui

   Solution. $\boxed{281}$

   We use geometric probability based on when Samuel starts to be disruptive.

   The viable area is a 80 minute by 80 minute square with the diagonal strip in the center omitted so that the two 10 minute periods do not overlap with an area of 70^2.

   The favorable area is the viable area without the area where the x or the y coordinates are between 35 and 60, because that is when Ada is doing the challenge problem.

   Calculating the probability, we find $\frac{2\cdot(20\cdot 35)+(35-10)^2+(20-10)^2}{70^2}=\frac{85}{196}$.

   Thus, $m=85, n=196$, and $m+n=85+196=\boxed{281}$. □

6. **[15]** Circle ω has radius 10 with center O. Let P be a point such that $PO=6$. Let the midpoints of all chords of ω through P bound a region of area R. Find the value of $\lfloor 10R\rfloor$.

   Proposed by Andrew Zhao

   Solution. $\boxed{282}$

   Let the midpoint of any given chord be M. We have $\angle PMO=90^\circ$, so the circumcircle of $\triangle PMO$ is a circle with diameter PO. Hence, all points M lie on this circle, and the area of this region is $\pi\left(\frac{PO}{2}\right)^2=9\pi$, so the answer is $\boxed{282}$. □

7. **[15]** Let S denote the sum of all rational numbers of the form $\frac{a}{b}$, where a and b are relatively prime positive divisors of 1300. If S can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers, then find $m+n$.

   Proposed by Ephram Chun

   Solution. $\boxed{2819}$

   The sum is equivalent to $\sum_{i|1300^2}\frac{i}{1300}$ when i is all the factors of 1300^2 because when $\frac{i}{1300^2}$ is reduced it will be a reduced fraction and both numbers will be relatively prime to each other and be factors of 1300. Since it a can be greater or less than b we need the factors of 1300^2. Thus our answer is

   $\frac{(1+13)(1+2+4)(1+5+25)}{1300}=\frac{14*7*31}{1300}=\frac{7*7*31}{1300}=\boxed{\frac{1519}{1300}}$ We are looking for $m+n$, which is $1519+1300=\boxed{2819}$. □

8. **[15]** Find the sum of all positive integers a such that there exists an integer n that satisfies the equation:

$$a! \cdot 2^{\lfloor \sqrt{a} \rfloor} = n!.$$

Proposed by Ivy Zheng

Solution. $\boxed{32}$

Note a must be one less than $2^{\lfloor \sqrt{a} \rfloor} = n$, and $n = a+1$. This means $a = 2^{\lfloor \sqrt{a} \rfloor} - 1$. Listing first few powers of 2, we see that only $a = 1, 31$ work, so we have $1 + 31 = \boxed{32}$. □

9. **[20]** $\triangle ABC$ has a right angle at B, $AB = 12$, and $BC = 16$. Let M be the midpoint of AC. Let ω_1 be the incircle of $\triangle ABM$ and ω_2 be the incircle of $\triangle BCM$. The line externally tangent to ω_1 and ω_2 that is not AC intersects AB and BC at X and Y, respectively. If the area of $\triangle BXY$ can be expressed as $\frac{m}{n}$, compute is $m+n$.

Proposed by Alex Li

Solution. $\boxed{1277}$

First, we note that letting BM and XY hit at P, then we have that the tangent from P to ω_1 has the same length of that of M to ω_2 and vice versa. Using the tangent segments, we see that PM has the same length as an external tangent, which has length $20 - \frac{12}{2} - \frac{16}{2} = 6$ so $PM = 6$ and $PB = 4$. Now, we see that ω_1 has a radius of 3 and ω_2 has a radius of $\frac{8}{3}$. Thus letting AC and XY hit at Q, then the external tangent segment with respect to the two circles has $\frac{1}{9}$ of the length of Q's tangent to ω_1 meaning that $PM = 9 \cdot 6 - 4 = 50$. Now applying Menalaus, we see that

$$1 = \frac{BP}{PM} \cdot \frac{MQ}{QC} \cdot \frac{CY}{YB} = \frac{4}{6} \cdot \frac{50}{40} \cdot \frac{16 - BY}{BY}$$

and substituting all values gets that $BY = \frac{80}{11}$. Similarly, we see that

$$1 = \frac{BP}{PM} \cdot \frac{MQ}{QA} \cdot \frac{AX}{XB} = \frac{4}{6} \cdot \frac{50}{60} \cdot \frac{12 - AY}{AY}$$

and substituting all values gets that $AY = \frac{30}{7}$. So, the area of $\triangle BXY$ is $\frac{1}{2} \cdot BX \cdot BY = \frac{1200}{77}$, and our answer is $1200 + 77 = \boxed{1277}$. □

10. **[20]** Define a sequence $\{a_n\}_{n \geq 1}$ recursively by $a_1 = 1$, $a_2 = 2$, and for all integers $n \geq 2$, $a_{n+1} = (n+1)^{a_n}$. Determine the number of integers k between 2 and 2020, inclusive, such that $k+1$ divides $a_k - 1$.

Proposed by Taiki Aiba

Solution. $\boxed{1009}$

The condition should hold if and only if k is odd. Note that $a_k = k^{a_{k-1}}$. It should be clear that a_k and k have the same parity, and $k \equiv -1 \pmod{k+1}$. Combining these two facts yields that $a_k \equiv (-1)^{k-1 \pmod 2}$. If k is even, then $a_k \equiv -1 \pmod{k+1}$, so $a_k - 1 \equiv -2 \pmod{k+1}$, and if k is odd, then $a_k \equiv 1 \pmod{k+1}$, so $a_k - 1 \equiv 0 \pmod{k+1}$, which is what we want. So, k must be odd for the conditions to hold. We count all of the odd numbers from 3 to 2019, inclusive, for a total of $\boxed{1009}$ numbers that work. □

11. **[20]** Two sequences of nonzero reals $a_1, a_2, a_3, \ldots$ and $b_2, b_3, \ldots$ are such that $b_n = \prod_{i=1}^{n} a_i$ and $a_n = \frac{b_n^2}{3b_n - 3}$ for all integers $n > 1$. Given that $a_1 = \frac{1}{2}$, find $|b_{60}|$.

Proposed by Andrew Zhao

Solution. $\boxed{3}$

$a_n = \frac{b_n}{b_{n-1}} \implies \frac{b_n}{b_{n-1}} = \frac{{b_n}^2}{3b_n - 3}$. From here we can divide by b_n because it's never 0, and get $\frac{1}{b_{n-1}} = \frac{b_n}{3b_n - 3}$. Rearranging gives $b_n = \frac{3}{3 - b_{n-1}}$, and then notice it has period 6. This means $b_{60} = b_6$, and calculating this gives us $b_6 = -3$, so our answer is $\boxed{3}$. □

12. **[20]** Richard comes across an infinite row of magic hats, $H_1, H_2, \ldots$ each of which may contain a dollar bill with probabilities $p_1, p_2, \ldots$. If Richard draws a dollar bill from H_i, then $p_{i+1} = p_i$, and if not, $p_{i+1} = \frac{1}{2}p_i$. If $p_1 = \frac{1}{2}$ and E is the expected amount of money Richard makes from all the hats, compute $\lfloor 100E \rfloor$.

    Proposed by Alex Li

    Solution. $\boxed{160}$

    Let E_k denote the expected value given that $p_1 = \frac{1}{2^k}$; we wish to find E_1, and we will proceed using recursion.

    Given $p_1 = \frac{1}{2^k}$, there is a $\frac{1}{2^k}$ probability that $p_2 = \frac{1}{2^k}$ and a $1 - \frac{1}{2^k}$ probability that $p_2 = \frac{1}{2^{k+1}}$. Thus we have

$$E_k = \frac{1}{2^k}(1 + E_k) + \left(1 - \frac{1}{2^k}\right) E_{k+1}.$$

    Simplifying the expression, we see that

$$E_{k+1} = E_k - \frac{1}{2^k - 1}.$$

    Let $E_\infty = \lim_{k\to\infty} E_k$. Note that

$$E_\infty = E_1 - \sum_{k=1}^{\infty} \frac{1}{2^k - 1}.$$

    It is clear that $E_\infty = 0$, which means

$$E_1 = \sum_{k=1}^{\infty} \frac{1}{2^k - 1}.$$

    This sum cannot be written in a closed form, but evaluating the sum of the first few values, we see that $E_1 = 1.60\ldots$ and the answer is $\boxed{160}$. □

13. **[25]** Find the number of integers n from 1 to 2020 inclusive such that there exists a multiple of n that consists of only 5's.

    Proposed by Ephram Chun and Taiki Aiba

    Solution. $\boxed{970}$

    Claim: All positive integers have a multiple that consists of only 1's and 0's. Proof: Consider $n+1$ integers $1, 11, 111, 1111, \ldots 111\ldots1 (n+1 1s)$. When divided by n there are a total of $n+1$ remainders. Thus, due to the Pigeonhole Principle 2 of these remainders must be equal so their difference which will always be in the form $111\ldots000$ with $1s$ followed by $0s$ is divisible by n.

    Claim: If n is relatively prime to 10 then there exists a multiple that only consists of $1s$. Proof: Using the proof above we know that all positive numbers can be written as a multiple consisting only $1s$ and $0s$. Thus, if n is relatively prime to 10 and we divide out all powers of 10 possible then we will be left with an integer in the form of $111\ldots1$ that will still be divisible by n.

    Since our number will be in the form $555\ldots5$, we can see that any number relatively prime to 10 can be multiplied by 5 to achieve a multiple with consecutive 5s. Thus, we have two sets: the set $\mathscr{S}$ of numbers relatively prime to 10, and the set of numbers five times some number in $\mathscr{S}$. We proceed to count.

    Note that from 1 to 10, inclusive, there are 4 numbers relatively prime to 10. We may multiply this by $\frac{2020}{10} = 202$ to account for the range 1 to 2020, inclusive. This gives us 808 numbers. For the second set, we count again the number of numbers relatively prime to 10, except the range is now 1 to $\frac{2020}{5} = 404$, inclusive. From 1 to 400, there are $40 \cdot 4 = 160$ numbers that are relatively prime to 10, and the numbers 401 and 403 are also relatively prime to 10. Thus, our answer is $808 + 160 + 2 = \boxed{970}$. □

14. **[25]** Two points E and F are randomly chosen in the interior of unit square $ABCD$. Let the line through E parallel to AB hit AD at E_1, the line through E parallel to AD hit CD at E_2, the line through F parallel to AB hit BC at F_1, and the line through F parallel to BC hit AB at F_2. The expected value of the overlap of the areas of rectangles EE_1DE_2 and FF_1BF_2 can be written as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

    Proposed by Kevin Zhao

Solution. $\boxed{37}$

We first see that there is a $\frac{1}{4}$ chance that these two rectangles overlap at all. Now, in the event they overlap, by Linearity of Expectation, the expected value of the overlapping area is the square of the expected distance between the two randomly chosen points on a number line from 0 to 1.
Let's calculate the expected distance between such points. Note that we are summing two portions of areas for the expected value of the distance of two points on a number line given one. This value is, for such a value $0 \le k \le 1$, we have $\frac{k^2}{2} + \frac{(1-k)^2}{2} = k^2 - k + 0.5$. Let's see the area below this curve for $0 \le k \le 1$. The area, cutting into n pieces, is approximated by

$$\sum_{i=1}^{n} \frac{\left(\frac{i}{n}\right)^2 - \left(\frac{i}{n}\right) + \frac{1}{2}}{n} = \sum_{i=1}^{n} \frac{i^2}{n^3} - \sum_{i=1}^{n} \frac{i}{n^2} + \frac{1}{2} = \frac{n(n+1)(2n+1)}{6n^3} - \frac{n(n+1)}{2n^2} + \frac{1}{2} = \frac{n(n+1)(1-n)}{6n^3} + \frac{1}{2}$$

and now, we see that as n approaches ∞, then the area is more accurate, and $\frac{1}{n^a}$ for all $a \ge 0$ approaches 0. So, our area is

$$\frac{n(n+1)(1-n)}{6n^3} + \frac{1}{2} = \frac{1}{2} - \frac{1}{6} + \frac{1}{6n^2}$$

and $\frac{1}{6n^2}$ approaches 0 so our area under this curve is just $\frac{1}{2} - \frac{1}{6} = \frac{1}{3}$. This means that the expected value of the distance of two points on a number line is $\frac{1}{3}$, so the expected value of the area, by Linearity of Expectation, is $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$ assuming they overlap. So, because this is $\frac{1}{4}$ of the expected value of the overlap of the areas of rectangles EE_1DE_2 and FF_1BF_2, our such value is $\frac{1}{36}$ and so our answer is $1 + 36 = \boxed{37}$. □

15. **[25]** Let x satisfy $x^4 + x^3 + x^2 + x + 1 = 0$. Compute the value of $(5x + x^2)(5x^2 + x^4)(5x^3 + x^6)(5x^4 + x^8)$.

Proposed by Andrew Zhao

Solution. $\boxed{521}$

Note that x is a fifth root of unity. We can rewrite the expression as

$$(5x + x^2)(5x^4 + x^8)(5x^2 + x^4)(5x^3 + x^6)$$

$$= (25x^5 + x^{10} + 5x^6 + 5x^9)(25x^5 + x^{10} + 5x^7 + 5x^8)$$

$$= (26 + 5(x + x^4))(26 + 5(x^2 + x^3))$$

$$= 676 + 130(x + x^4) + 130(x^2 + x^3) + 25(x + x^4)(x^2 + x^3)$$

$$= 676 + 130(x + x^2 + x^3 + x^4) + 25(x^3 + x^4 + x^6 + x^7)$$

$$= 676 + 155(x + x^2 + x^3 + x^4).$$

Note that in the equation $n^5 - 1 = 0$, the solutions are x, x^2, x^3, x^4, and x^5. Hence, by Vieta's, $x + x^2 + x^3 + x^4 = 0 - x^5 = -1$, so our answer is $676 - 155 = \boxed{521}$. □

16. **[25]** Two circles ω_1 and ω_2 have centers O_1 and O_2, respectively, and intersect at points M and N. The radii of ω_1 and ω_2 are 12 and 15, respectively, and $O_1O_2 = 18$. A point X is chosen on segment MN. Line O_1X intersects ω_2 at points A and C, where A is inside ω_1. Similarly, line O_2X intersects ω_1 at points B and D, where B is inside ω_2. The perpendicular bisectors of segments AB and CD intersect at point P. Given that $PO_1 = 30$, find PO_2^2.

Proposed by Andrew Zhao

Solution. $\boxed{981}$

We have $\text{Pow}_{\omega_1}(X) = (XB)(XD)$ and $\text{Pow}_{\omega_2}(X) = (XA)(XC)$. Because X lies on the radical axis of ω_1 and ω_2, we have $(XB)(XD) = (XA)(XC)$. Thus, by the Converse of Power of a Point, $ABCD$ is cyclic. Let the circumcircle of $ABCD$ be ω_3. We then have that P, the intersection of the perpendicular bisectors of two sides of $ABCD$, must be the center of ω_3. The line connecting the centers of two circles must be perpendicular to their radical axis, so $O_1C \perp PO_2$, and $O_2D \perp PO_1$. This means that X, the intersection of O_1C and O_2D, is the orthocenter of $\triangle PO_1O_2$. Therefore, because MN is perpendicular to O_1O_2 and passes through X, P lies on MN. (In other words, consider drawing all altitudes of $\triangle PO_1O_2$.) Hence, $\text{Pow}_{\omega_1}(P) = \text{Pow}_{\omega_2}(P) \implies PO_1^2 - 12^2 = PO_2^2 - 15^2$, which leads to $PO_2^2 = \boxed{981}$. □

17. **[30]** There are n ordered tuples of positive integers (a, b, c, d) that satisfy

$$a^2 + b^2 + c^2 + d^2 = 13 \cdot 2^{13}.$$

Let these ordered tuples be $(a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2), \ldots, (a_n, b_n, c_n, d_n)$. Compute $\sum_{i=1}^{n}(a_i + b_i + c_i + d_i)$.

Proposed by Kaylee Ji

Solution. $\boxed{3840}$

Since $13 \cdot 2^{13} \equiv 0 \pmod 4$, and perfect squares are always 0 or 1, modulo 4, a, b, c, and d must all be odd or all be even. If a, b, c, and d are odd then $a^2 + b^2 + c^2 + d^2 \equiv 4 \pmod 8$, but $13 \cdot 2^{13} \equiv 0 \pmod 8$. Therefore, a, b, c, and d are all even. We let $a = 2a_1, b = 2b_1$, $c = 2c_1$, and $d = 2d_1$.

Then $13 \cdot 2^{13} = a^2 + b^2 + c^2 + d^2 = 4(a_1^2 + b_1^2 + c_1^2 + d_1^2)$. Dividing both sides of the equation by 4, we get $a_1^2 + b_1^2 + c_1^2 + d_1^2 = 13 \cdot 2^{11}$.

We continue this process until we end up with $a_6^2 + b_6^2 + c_6^2 + d_6^2 = 13 \cdot 2$. The only solutions to $a_6^2 + b_6^2 + c_6^2 + d_6^2 = 26$ in positive integers are $(3, 3, 2, 2)$ and permutations. and $a = 64a_6$, $b = 64b_6$, $c = 64c_6$, and $d = 64d_6$. Therefore, the only solutions in positive integers to $a^2 + b^2 + c^2 + d^2 = 13 \cdot 2^{13}$ are $(192, 192, 128, 128)$ and permutations. There are 6 permutations and each sum up to 640. Therefore, we have $6 \cdot 640 = \boxed{3840}$. □

18. **[30]** Let f of degree at most 13 such that $f(k) = 13^k$ for $0 \le k \le 13$. Compute the last three digits of $f(14)$.

Proposed by Kaylee Ji

Solution. $\boxed{425}$

We claim that the polynomial can be written as

$$f(x) = \sum_{i=0}^{13} 13^i \prod_{0 \le j \le 13, i \ne j} \frac{x - j}{i - j}.$$

Each term in the expansion of $f(x) = \sum_{i=0}^{13} 13^i \prod_{0 \le j \le 13, i \ne j} \frac{x-j}{i-j}$ has degree of at most 13 since each $\prod_{0 \le j \le 13, i \ne j} \frac{x-j}{i-j}$ is the product of 13 linear polynomials which results in a polynomial of degree 13. Therefore, the sum of these products can have degree of at most 13.

Note that the fourteen given values of f uniquely define the polynomial. Now we just need to confirm $f(k) = 13^k$ for $0 \le k \le 13$. $f(k) = \sum_{i=0}^{13} 13^i \prod_{0 \le j \le 13, i \ne j} \frac{k-j}{i-j}$.

When $i \ne k$, $\prod_{0 \le j \le 13, i \ne j} \frac{k-j}{i-j} = 0$ since j can equal to k so the product contains the term $k - k = 0$.

When $i = k$, $\prod_{0 \le j \le 13, i \ne j} \frac{k-j}{i-j} = \prod_{0 \le j \le 13, i \ne j} \frac{k-j}{k-j} = 1$.

Therefore, $f(k) = \sum_{0 \le i \le 13, i \ne k} 0 + 13^k \cdot 1 = 13^k$. Then,

$$f(14) = \sum_{i=0}^{13} 13^i \prod_{0 \le j \le 13, i \ne j} \frac{14 - j}{i - j}$$

$$f(14) = \sum_{i=0}^{13} 13^i \binom{14}{i} (-1)^{13-i}$$

$$f(14) = -(13 - 1)^{14} + 13^{14} = 13^{14} - 12^{14}.$$

To find the last 3 digits of this, we can use a four function calculator over and over(truncate to leave just last three digits if number gets two big), and we get 425. □

19. **[30]** Euhan and Minjune are playing a game. They choose a number N so that they can only say integers up to N. Euhan starts by saying the 1, and each player takes turns saying either $n + 1$ or $4n$ (if possible), where n is the last number said. The player who says N wins. What is the smallest number larger than 2019 for which Minjune has a winning strategy?

Proposed by Janabel Xia

Solution. $\boxed{2048}$

Euhan is guaranteed all odd N, because he can just say $n+1$ each time and Minjune will be stuck saying even numbers (and they will eventually reach N). Now we consider even N.

Claim: If player X can guarantee winning $N = k$, they can guarantee winning if N is even and between $16k$ and $16k+14$ inclusive.

Proof: Basically, each player wants to guarantee that they quadruple at some turn such that no one can quadruple after that. Then that player will just say the remaining evens up to N. If player X can say k (and N is an even between $16k$ and $16k+14$ inclusive), then if player Y says $4k$, player X can just say $16k$ and clearly win, and if player Y says $k+1$, player X can say $4k+4$ and no one can quadruple anymore. If player X can't say k, then player Y must have first said a number $m > k$. Now m must be between $k+1$ and $4k-1$, since it is the first number said larger than k and one can't skip this interval, as the largest move is quadrupling. Then player X can just quadruple to get an even between $4k+4$ and $16k-4$ inclusive, and this will be the last quadruple possible, so player X wins.

Now it's not hard to see that Minjune can generate (and win) any N with the process above that has only even digits in base 16, and Euhan wins the complement of this set; if N has an odd digit in base 16, strip away the last digits by going backwards in the process above and then you get an odd number, which Euhan can win, so by generating forwards, Euhan can win the original number. Hence the smallest N greater than 2019 that Minjune can win is $\boxed{2048}$. □

20. **[30]** Let $ABCD$ be a cyclic quadrilateral with center O with $AB > CD$ and $BC > AD$. Let M and N be the midpoint of sides AD and BC, respectively, and let X and Y be on AB and CD, respectively, such that $AX \cdot CY = BX \cdot DY = 20000$, and $AX \le CY$. Let lines AD and BC hit at P, and let lines AB and CD hit at Q. The circumcircles of $\triangle MNP$ and $\triangle XYQ$ hit at a point R that is on the opposite side of CD as O. Let R_1 be the midpoint of PQ and B, D, and R be collinear. Let O_1 be the circumcenter of $\triangle BPQ$. Let the lines BO_1 and DR_1 intersect at a point I. If $BP \cdot BQ = 823875$, $AB = 429$, and $BC = 495$, then $IR = \frac{a\sqrt{b}}{c}$ where a, b, and c are positive integers, b is not divisible by the square of a prime, and $\gcd(a, c) = 1$. Find the value of $a + b + c$.

Proposed by Kevin Zhao

Solution. $\boxed{550}$

First, we note that $\frac{AX}{XB} = \frac{DY}{YC}$. Now, let R' be the Miquel point of quadrilateral $ABCD$ and points P and Q. We see that $\triangle R'CD \sim \triangle R'BA$ which maps X to Y. Similarly, $\triangle R'AD \sim \triangle R'BC$ so since we have such a spiral similarity, then M maps to N, too, and $R = R'$. So, since MD and NC meet at P, then $PRMN$ is cyclic. Similarly, we see that $QRYX$ is cyclic.

Now, we see that B, D, and R are collinear, so $\angle BDC + \angle CDR = 180^\circ$ and since $\angle BAC = \angle BDC$ and $\angle CDR = 180^\circ - \angle RPC$ then $\angle RPC = \angle QPB = \angle BAC$ so $\triangle BAC \sim \triangle BPQ$. In addition, note $\angle PRB = \angle PAB$ so $\angle QRD = \angle DAQ$ but since $QRDA$ is cyclic, then $\angle PRD = \angle DAQ$ so D is the orthocenter of $\triangle BPQ$.

Now, note that BO_1 and DR_1 thus hit at a point I on the circumcircle of $\triangle BPQ$. I is the antipode of A with respect to the circumcircle of $\triangle BPQ$ and also the reflection of D. We also note that $ACPQ$ is a cyclic quadrilateral.

By Power of a Point, $BA \cdot BQ = BC \cdot BP$ so $429 \cdot BQ = 495 \cdot BP$. Since $\frac{BP}{BQ} = \frac{BA}{BC} = \frac{429}{495} = \frac{13}{15}$, then because $BP \cdot BQ = 823875$, then $BP^2 = \frac{13}{15} \cdot 823875 = 714025$ and so $BP = \sqrt{714025} = 845$ and $BQ = \frac{15}{13} \cdot BP = 975$. This means that $\cos(\angle BAC) = \frac{429}{845}$ and Law of Cosines shows that

$$PQ = \sqrt{BP^2 + BQ^2 - 2 \cdot BP \cdot BQ \cdot \cos(\angle BAC)} = \sqrt{845^2 + 975^2 - 2 \cdot 975 \cdot 429} = \sqrt{828100} = 910$$

and so the ratio of $BP : PQ : QB$ is $13 : 14 : 15$. Now, as a result of that, we know $BR = \frac{6}{7} \cdot PQ = 780$, $PR = \frac{5}{14} \cdot PQ = 325$, and so since $PR_1 = \frac{PQ}{2} = 455$, then $RR_1 = 130$. Note that $BD = \frac{13}{12} \cdot BC = \frac{2145}{4}$ so $DR = BR - BD = \frac{975}{4}$ and

$$IR = \sqrt{DR^2 + (2RR_1)^2} = \frac{\sqrt{975^2 + 1040^2}}{4} = \frac{\sqrt{2032225}}{4} = \frac{65\sqrt{481}}{4}$$

which means that our answer is $65 + 481 + 4 = \boxed{550}$. □

Among Us

21. **[15]** The LHS Math Team wants to play Among Us. There are so many people who want to play that they are going to form several games. Each game has at most 10 people. People are *happy* if they are in a game that has at least 8 people in it. What is the largest possible number of people who would like to play Among Us such that it is impossible to make everyone *happy*?

Proposed by Samuel Charney

Solution. $\boxed{31}$

This is equivalent to finding the largest number that cannot be formed by adding multiples of 8, 9, and 10. Considering these sums modulo 8, it is clear that the hardest remainder to achieve mod 8 is 7, with the smallest achievable being 39. Therefore, the largest number of people such that not everyone is happy is $39-8=\boxed{31}$. □

22. **[20]** In a game of Among Us, there are 10 players and 12 colors. Each player has a "default" color that they will automatically get if nobody else has that color. Otherwise, they get a random color that is not selected. If 10 random players with random default colors join a game one by one, the expected number of players to get their default color can be expressed as $\frac{m}{n}$. Compute $m+n$. Note that the default colors are not necessarily distinct.

Proposed by Jeff Lin

Solution. $\boxed{29}$

We realize that the first person is guaranteed to get their default color, the second person has a $\frac{11}{12}$ chance, the third person has a $\frac{10}{12}$ chance, and so on. Thus, the answer is just $1+\frac{11}{12}+\frac{10}{12}+\ldots\frac{3}{12}=\frac{75}{12}=\frac{25}{4}$. $25+4=\boxed{29}$. □

23. **[20]** There are 5 people left in a game of Among Us, 4 of whom are crewmates and the last is the impostor. None of the crewmates know who the impostor is. The person with the most votes is ejected, unless there is a tie in which case no one is ejected. Each of the 5 remaining players randomly votes for someone other than themselves. The probability the impostor is ejected can be expressed as $\frac{m}{n}$. Find $m+n$.

Proposed by Samuel Charney

Solution. $\boxed{145}$

Let's consider each case that results in the impostor being ejected. All four crewmates vote for the impostor with probability $\frac{1}{256}$, and who the impostor votes for does not matter. If three impostors vote for the crewmate, there are 4 choices for which crewmate doesn't, and 3 choices for who that person votes for. Who the impostor votes for does not matter, so the probability is $\frac{12}{256}$. If only two crewmates vote for the impostor, the 3 remaining players must all vote for different people. There are 6 ways to select the pair of crewmates who vote for the impostor. The two other crewmates can vote for different people, other than the impostor, in 7 ways, and the impostor has 2 remaining choices. Therefore, the probability is $\frac{84}{1024}$. The answer is $\frac{1}{256}+\frac{12}{256}+\frac{84}{1024}=\frac{17}{128}$, which gives $17+128=\boxed{145}$. □

24. **[25]** Sam has 1 Among Us task left. He and his task are located at two randomly chosen distinct vertices of a 2021-dimensional unit hypercube. Let E denote the expected distance he has to walk to get to his task, given that he is only allowed to walk along edges of the hypercube. Compute $\lceil 10E \rceil$.

Proposed by Samuel Charney

Solution. $\boxed{10106}$

Let Sam be at a fixed point. We first claim number of paths with distance i is $\binom{2021}{i}$. To prove this we use induction. It is easy to see for a square there are 2 points 1 unit away and 1 point 2 units away. If this is true for n dimensions, then for $n+1$ dimensions we can either stay in n dimensions or move $i-1$ times in n dimensions with 1 jump into the higher dimension, noting there is exactly 1 or else we would be moving away if this jump happens again. This gives us $\binom{n}{i}+\binom{n}{i-1}=\binom{n+1}{i}$ as desired. Now we must compute

$$\sum_{1\le i\le 2021}\binom{2021}{i}i$$

Which is the same as

$$\sum_{0\le i\le 2021}\binom{2021}{i}i$$

For $i=x$ and $i=2021-x$, the partial sum is $\binom{2021}{x}x+\binom{2021}{2021-x}(2021-x)=\binom{2021}{x}x+\binom{2021}{x}(2021-x)=\binom{2021}{x}2021$. Therefore, we must compute

$$\frac{1}{2}\sum_{1\le i\le 2021}\binom{2021}{i}2021=\frac{2021}{2}\sum_{1\le i\le 2021}\binom{2021}{i}=\frac{2021}{2}\cdot 2^{2021}=2021*2^{2020}$$

There are $2^{2021}-1$ points where the second task could be. Therefore, $E=\frac{2021\cdot 2^{2020}}{2^{2021}-1}$ so $10E=\frac{10105\cdot 2^{2021}}{2^{2021}-1}=10105\cdot\frac{2^{2021}}{2^{2021}-1}$. $\frac{2^{2021}}{2^{2021}-1}$ is very slightly greater than 1, so the answer is $\boxed{10106}$. □

Radishes

25. **[15]** Alex and Kevin are radish watching. The probability that they will see a radish within the next hour is $\frac{1}{17}$. If the probability that they will see a radish within the next 15 minutes is p, determine $\lfloor 1000p\rfloor$. Assume that the probability of seeing a radish at any given moment is uniform for the entire hour.

Proposed by Ephram Chun

Solution. $\boxed{492}$

Let the probability of seeing a radish in the next 15 minutes be p.

$$p^4=\frac{1}{17}$$

$$p\approx 0.49247$$

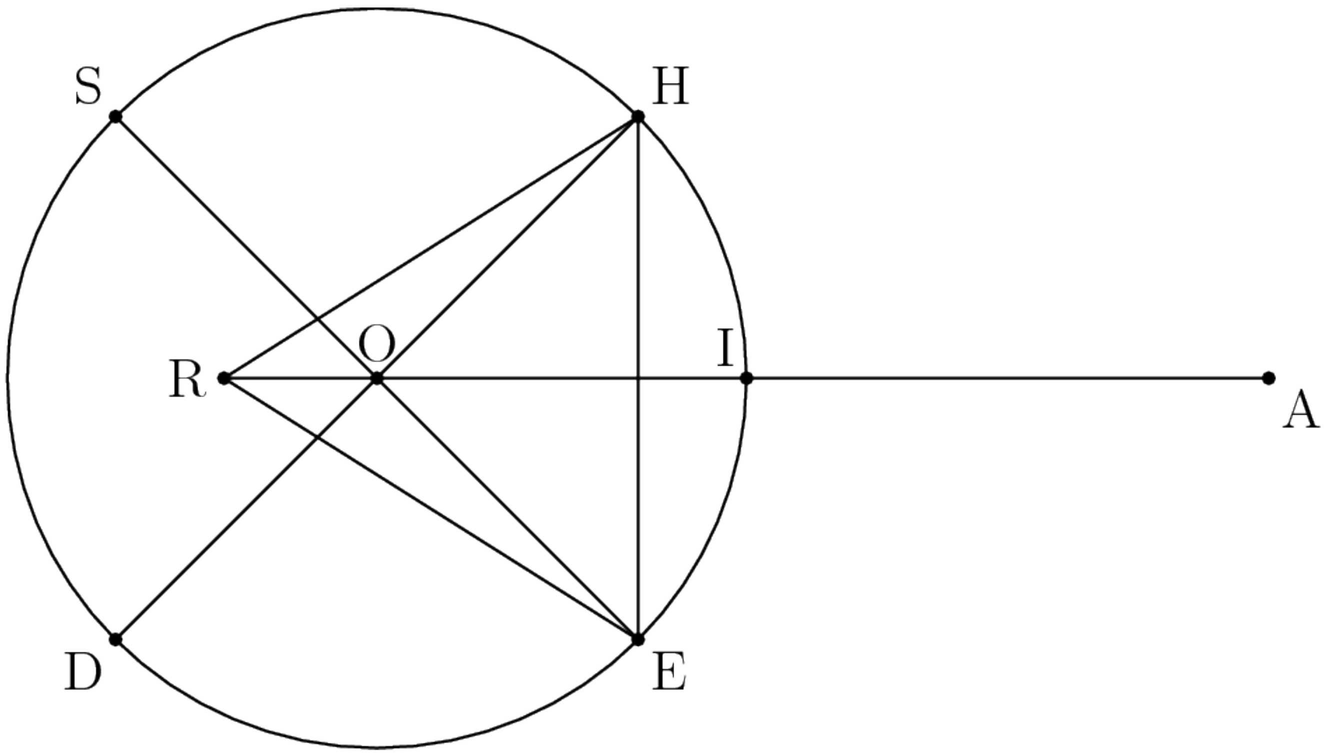
$$\lfloor 1000p\rfloor=\lfloor 1000*0.49247\rfloor=\boxed{492}.$$

□

26. **[20]** Jeff has planted 7 radishes, labelled R, A, D, I, S, H, and E. Taiki then draws circles through S,H,I,E,D, then through E,A,R,S, and then through H,A,R,D, and notices that lines drawn through SH, AR, and ED are parallel, with $SH=ED$. Additionally, HER is equilateral, and I is the midpoint of AR. Given that $HD=2$, HE can be written as $\frac{-\sqrt{a}+\sqrt{b}+\sqrt{1+\sqrt{c}}}{2}$, where a,b, and c are integers, find $a+b+c$.

Proposed by Jeff Lin

Solution. $\boxed{17}$



Since we are given 3 cyclic quadrilaterals, we can use radical axis theorem to get that HD, ES, and AR are concurrent at a radical center O. We also have $OH * OD = OE * OS = OA * OR$. Since SH and ED are parallel and $SHED$ is cyclic, we have $SHED$ is an isoceles trapezoid, with O being the intersection of SE and HD, and OR parallel to the sides of $SHED$. Now, it is easy to see that $SHED$ must be a rectangle, as $RE = RH$. Therefore, O is the center of $SHIED$ and $OE = 1$.

Since O is also a radical center, we have $RO * OA = 1$, but I is the midpoint and $IO = 1$, so $RO * (RO + 2) = 1$, or $RO = \sqrt{2} - 1$. Now, we can use Law of Cosines to compute $HE = HR$. We have $\angle ORH = 30$, so

$$HR^2 + (\sqrt{2}-1)^2 - \sqrt{3} * HR * (\sqrt{2}-1) = 1$$

$$HR^2 + 3 - 2\sqrt{2} - (\sqrt{6}-\sqrt{3})HR = 1$$

Solving this as a quadratic in HR gives

$$HR = \frac{\sqrt{6}-\sqrt{3}+\sqrt{(\sqrt{6}-\sqrt{3})^2 - 4*(2-2\sqrt{2})}}{2} = \frac{-\sqrt{3}+\sqrt{6}+\sqrt{1+2\sqrt{2}}}{2}$$

Putting this in the form we want gives $a = 3, b = 6$, and $c = 8$, so the answer is $3 + 6 + 8 = \boxed{17}$ □

27. **[25]** Ephram is growing 3 different variants of radishes in a row of 13 radishes total, but he forgot where he planted each radish variant and he can't tell what variant a radish is before he picks it. Ephram knows that he planted at least one of each radish variant, and all radishes of one variant will form a consecutive string, with all such possibilities having an equal chance of occurring. He wants to pick three radishes to bring to the farmers market, and wants them to all be of different variants. Given that he uses optimal strategy, the probability that he achieves this can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Proposed by Jeff Lin

Solution. $\boxed{17}$

Considering the possible distributions of radish variants, we notice that the possibilities correspond to choosing 2 of the 12 spots between the radishes to act as "variant dividers". Thus, there are 66 total possibilities. We also notice that the first and last radish must be of different variants, and it is likely optimal to choose both of them. If we choose the first, last, and middle radish, the cases where all three are different are when one divider is to the left of the middle and one is to the right, giving 36 cases. Thus, we get $\frac{6}{11}$, which gives $6 + 11 = \boxed{17}$. □

COVID

28. **[15]** Arthur has a regular 11-gon. He labels the vertices with the letters in $CORONAVIRUS$ in consecutive order. Every non-ordered set of 3 letters that forms an isosceles triangle is a member of a set S, i.e. {C, O, R} is in S. How many elements are in S?

Proposed by Samuel Charney

Solution. $\boxed{53}$

We need to remove the number of double counted sets from the total number of sets. The total number of isosceles triangles is $11 \cdot 5 = 55$ because you pick the top vertex out of 11 vertices and then have 5 choices for the height. A set is double counted if you can use the other O and or the other R to still get an isosceles triangle. If two isosceles triangles share a side in common and have each O as their third vertex, the four points must form a parallelogram. However this is impossible as the total number of points is not a multiple of 4. The same is true for switching with R is included. Now for switching the O and the R. The isosceles triangles with R_1, not R_2, and only one of the Os are CO_1R_1, R_1O_2N, and O_1R_1I. CO_2R_2 and R_2O_1N are both isosceles so we remove them from the count, while O_2R_2I is not. Therefore, we only have 2 double counted sets, for a final answer of $\boxed{53}$. □

29. **[20]** Find the smallest possible value of n such that $n+2$ people can stand inside or on the border of a regular n-gon with side length 6 feet where each pair of people are at least 6 feet apart.

Proposed by Jeff Lin

Solution. $\boxed{9}$

The answer is $\boxed{9}$. It is easy to see why this does not work for $n = 8$. For 9, we can place a person at each vertex, and then draw a circle with radius 6 centered at each vertex. Then, I claim that we can put a person on two intersection points that are semi-opposite each other. To see this, let the nonagon be labelled ABCDEFGHI. Then, let a 10th person be standing on a point J that is exactly 6 feet from both A and B. Then, $\angle ABJ = 60$. Since angles in a nonagon have angles of 140, we see that $\angle JBC = 80$. However, another person standing at K 6 feet from both E and F would have $\angle KED. = 80$. Now, consider the isoceles trapezoid $BCDE$. This trapezoid has 2 angles of 140, so the other 2 must be 40. Since $80 = 2 * 40$, $BJKD$ must be the reflection of $BCDE$ over BE. Thus, $JK = 6$, so they are at least 6 feet apart, as desired. □

30. **[25]** A large gathering of people stand in a triangular array with 2020 rows, such that the first row has 1 person, the second row has 2 people, and so on. Every day, the people in each row infect all of the people adjacent to them in their own row. Additionally, the people at the ends of each row infect the people at the ends of the rows in front of and behind them that are at the same side of the row as they are. Given that two people are chosen at random to be infected with COVID at the beginning of day 1, what is the earliest possible day that the last uninfected person will be infected with COVID?

Proposed by Richard Chen

Solution. $\boxed{1346}$

We claim that one person at the 1346th row on the very left end and one at the bottom row corner on the very right end would do the trick.

The first key observation is that having our two infected people on the ends of the rows is optimal. If we did put someone inside a row, COVID would need to first spread to the end of the row, and then begin to propagate to other rows. Another important advantage is that if someone is on the nth row, it takes $n-1$ days for them to fully infect all the people from rows 1 to n.

We can also reason that we want one person on the left side, at row j, and one person on the right side, at row k; without loss of generality, let $j \le k$. Now, our main challenge is: try to fill up the rows from j to 2020 in the exact same number of days as we need to fill up rows 1 to j(which we already know to take $j-1$ days).

We now claim that $k = 2020$ is best. The path between our two people going through row l for $j < l < k$ actually has length $k + l - j - 2$, while the path between our two people going through row l for $j < k < l$ has length $3l - j - k - 2$.

Notice that $k+l-j-2<3l-j-k-2 \implies 2k<2l$ always holds true; we actually find out that the larger l is, the longer the path is, so when $l=2020$ is the longest journey for all k. Thus, having $k=2020$ is best because then the path through $l=2020$ is shortest.

After finding $k=2020$ is best, we now just need to equate $j-1$ with $\frac{1}{2}\cdot(4040-j-2)$. This gives us j equal to $\frac{4040}{3}$. Checking the two integers closest to that, we see that both $j=1346, 1347$ gives a time of $\boxed{1346}$ days minimum.

□