LMT Fall Online: Division B

December 5th – December 12th, 2020

Contest Instructions

Contest Window

The Team Round consists of 30 short answer problems, including 10 theme problems. All answers are non-negative integers. The problems will be made available on the homepage of the LMT website on **Saturday, December 5th, at 3:00 pm**. Teams will have until **Saturday, December 12th, at 3:00 pm** to submit their answers using the link provided by email.

Contest Rules

With the exception of **standard four-function calculators**, computational aids including but not limited to scientific and graphing calculators, computer programs, and software such as Geogebra, Mathematica, and WolframAlpha, are **not** allowed. Communication of any form between students on different teams is similarly prohibited, and any team caught either giving or receiving an unfair advantage over other competitors will be disqualified. What constitutes cheating will be up to the final discretion of the competition organizers, who reserve the right to disqualify any team suspected of violating these rules.

Submitting Answers and Editing Team Information

During registration, your captain will be emailed a link to your team's homepage. This is where you will be able to update team information and answer submissions. We recommend that your captain distribute this link to the rest of the team so that the entire team has access. Once on your team's homepage, to enter or edit team answers, click the link next to "Submission Link:". Team name, team member names, and grades may be edited through the homepage as well. Remember, team member names must be the real names of the people on your team, and the team name must be appropriate.

Errata

If you believe there to be an error in one of the questions, email us at lmt@lhsmath.org with "Clarification" as the subject. Clarifications for problems will be updated on the LMT homepage, if necessary.

Scoring

The score of your team is the sum of the point values of the problems you answered correctly. Note that the theme problems deviate from the general trend in point values. Ties will not be broken. Results will be posted shortly after the competition, and the top teams will be recognized.

AoPS Art of Problem Solving
RSM Russian School of Mathematics
AwesomeMath®

1. **[10]** Four Ls are equivalent to three Ms. Nine Ms are equivalent to fourteen Ts. Seven Ts are equivalent to two Ws. If Kevin has thirty-six Ls, how many Ws would that be equivalent to?

2. **[10]** The area of a square is 144. An equilateral triangle has the same perimeter as the square. The area of a regular hexagon is 6 times the area of the equilateral triangle. What is the perimeter of the hexagon?

3. **[10]** Find the number of ways to arrange the letters in $LEXINGTON$ such that the string LEX does not appear.

4. **[10]** Find the greatest prime factor of $20!+20!+21!$.

5. **[15]** Given the following system of equations

$$\begin{aligned} a_1+a_2+a_3&=1\\ a_2+a_3+a_4&=2\\ a_3+a_4+a_5&=3\\ &\vdots\\ a_{12}+a_{13}+a_{14}&=12\\ a_{13}+a_{14}+a_1&=13\\ a_{14}+a_1+a_2&=14, \end{aligned}$$

find the value of a_{14}.

6. **[15]** 1001 marbles are drawn at random and without replacement from a jar of 2020 red marbles and n blue marbles. Find the smallest positive integer n such that the probability that there are more blue marbles chosen than red marbles is strictly greater than $\frac{1}{2}$.

7. **[15]** Zachary tries to simplify the fraction $\frac{2020}{5050}$ by dividing the numerator and denominator by the same integer to get the fraction $\frac{m}{n}$, where m and n are both positive integers. Find the sum of the (not necessarily distinct) prime factors of the sum of all the possible values of $m+n$.

8. **[15]** In rectangle $ABCD$, $AB=3$ and $BC=4$. If the feet of the perpendiculars from B and D to AC are X and Y, the length of XY can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

9. **[20]** Ben writes the string

$$\underbrace{111\dots11}_{\text{2020 digits}}$$

on a blank piece of paper. Next, in between every two consecutive digits, he inserts either a plus sign (+) or a multiplication sign (×). He then computes the expression using standard order of operations. Find the number of possible distinct values that Ben could have as a result.

10. **[20]** In a certain Zoom meeting, there are 4 students. How many ways are there to split them into any number of distinguishable breakout rooms, each with at least 1 student?

11. **[20]** $\triangle ABC$ is an isosceles triangle with $AB=AC$. Let M be the midpoint of BC and E be the point on AC such that $AE:CE=5:3$. Let X be the intersection of BE and AM. Given that the area of $\triangle CMX$ is 15, find the area of $\triangle ABC$.

12. **[20]** Find the sum of all positive integers a such that there exists an integer n that satisfies the equation:

$$a!\cdot 2^{\lfloor\sqrt{a}\rfloor}=n!.$$

13. **[25]** Compute the number of ways there are to completely fill a 3×15 rectangle with non-overlapping 1×3 rectangles.

14. **[25]** Let $\triangle ABC$ with $AB=AC$ and $BC=14$ be inscribed in a circle ω. Let D be the point on ray BC such that $CD=6$. Let the intersection of AD and ω be E. Given that $AE=7$, find AC^2.

15. **[25]** Let S denote the sum of all rational numbers of the form $\frac{a}{b}$, where a and b are relatively prime positive divisors of 1300. If S can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers, then find $m+n$.

16. **[25]** Let f be a function $\mathbb{R} \to \mathbb{R}$ that satisfies the following equation:

$$f(x)^2 + f(y)^2 = f(x^2 + y^2) + f(0).$$

If there are n possibilities for the function, find the sum of all values of $n \cdot f(12)$.

17. **[30]** Circle ω has radius 10 with center O. Let P be a point such that $PO = 6$. Let the midpoints of all chords of ω through P bound a region of area R. Find the value of $\lfloor 10R \rfloor$.

18. **[30]** Define a sequence $\{a_n\}_{n\geq 1}$ recursively by $a_1 = 1$, $a_2 = 2$, and for all integers $n \geq 2$, $a_{n+1} = (n+1)^{a_n}$. Determine the number of integers k between 2 and 2020, inclusive, such that $k+1$ divides $a_k - 1$.

19. **[30]** Ada is taking a math test from 12:00 to 1:30, but her brother, Samuel, will be disruptive for two ten-minute periods during the test. If the probability that her brother is not disruptive while she is solving the challenge problem from 12:45 to 1:00 can be expressed as $\frac{m}{n}$, find $m+n$.

20. **[30]** Two sequences of nonzero reals $a_1, a_2, a_3, \ldots$ and $b_2, b_3, \ldots$ are such that $b_n = \prod_{i=1}^{n} a_i$ and $a_n = \frac{b_n^2}{3b_n - 3}$ for all integers $n > 1$. Given that $a_1 = \frac{1}{2}$, find $|b_{60}|$.

Among Us

21. **[15]** Let $\triangle AMO$ be an equilateral triangle. Let U and G lie on side AM, and let S and N lie on side AO such that $AU = UG = GM$ and $AS = SN = NO$. Find the value of $\frac{[MONG]}{[USA]}$.

22. **[20]** A cube has one of its vertices and all edges connected to that vertex deleted. How many ways can the letters from the word "AMONGUS" be placed on the remaining vertices of the cube so that one can walk along the edges to spell out "AMONGUS"? Note that each vertex will have at most 1 letter, and one vertex is deleted and not included in the walk.

23. **[20]** The LHS Math Team wants to play Among Us. There are so many people who want to play that they are going to form several games. Each game has at most 10 people. People are *happy* if they are in a game that has at least 8 people in it. What is the largest possible number of people who would like to play Among Us such that it is impossible to make everyone *happy*?

24. **[25]** In a game of Among Us, there are 10 players and 12 colors. Each player has a "default" color that they will automatically get if nobody else has that color. Otherwise, they get a random color that is not selected. If 10 random players with random default colors join a game one by one, the expected number of players to get their default color can be expressed as $\frac{m}{n}$. Compute $m+n$. Note that the default colors are not necessarily distinct.

Radishes

25. **[15]** Emmy goes to buy radishes at the market. Radishes are sold in bundles of 3 for $5 and bundles of 5 for $7. What is the least number of dollars Emmy needs to buy exactly 100 radishes?

26. **[20]** Aidan owns a plot of land that is in the shape of a triangle with side lengths 5,10, and $5\sqrt{3}$ feet. Aidan wants to plant radishes such that there are no two radishes that are less than 1 foot apart. Determine the maximum number of radishes Aidan can plant.

27. **[25]** Alex and Kevin are radish watching. The probability that they will see a radish within the next hour is $\frac{1}{17}$. If the probability that they will see a radish within the next 15 minutes is p, determine $\lfloor 1000p \rfloor$. Assume that the probability of seeing a radish at any given moment is uniform for the entire hour.

COVID

28. **[15]** There are 2500 people in Lexington High School, who all start out healthy. After 1 day, 1 person becomes infected with coronavirus. Each subsequent day, there are twice as many newly infected people as on the previous day. How many days will it be until over half the school is infected?

29. **[20]** Alicia bought some number of disposable masks, of which she uses one per day. After she uses each of her masks, she throws out half of them (rounding up if necessary) and reuses each of the remaining masks, repeating this process until she runs out of masks. If her masks lasted her 222 days, how many masks did she start out with?

30. **[25]** Arthur has a regular 11-gon. He labels the vertices with the letters in $CORONAVIRUS$ in consecutive order. Every non-ordered set of 3 letters that forms an isosceles triangle is a member of a set S, i.e. {C, O, R} is in S. How many elements are in S?