Individual Round Solutions

Lexington High School

December 7th, 2019

1. For positive real numbers x, y, the operation $\otimes$ is given by $x \otimes y = \sqrt{x^2 - y}$ and the operation $\oplus$ is given by $x \oplus y = \sqrt{x^2 + y}$. Compute
$$(((5 \otimes 4) \oplus 3) \otimes 2) \oplus 1.$$

   Proposed by Ezra Erives

   Solution. $\boxed{\sqrt{23}}$

   Upon expansion, the expression evaluates to $\sqrt{25-4+3-2+1} = \boxed{\sqrt{23}}$. □

2. Janabel is cutting up a pizza for a party. She knows there will either be 4, 5, or 6 people at the party including herself, but can't remember which. What is the least number of slices Janabel can cut her pizza to guarantee that everyone at the party will be able to eat an equal number of slices?

   Proposed by Alex Li

   Solution. $\boxed{60}$

   The least common multiple of 4, 5, and 6 is $\boxed{60}$. □

3. If the numerator of a certain fraction is added to the numerator and the denominator, the result is $\frac{20}{19}$. What is the fraction?

   Proposed by Sammy Charney

   Solution. $\boxed{\dfrac{10}{9}}$

   We have $\frac{2a}{a+b} = \frac{20}{19}$, so $38a = 20a + 20b$, so $18a = 20b$, so the answer is $\frac{a}{b} = \boxed{\dfrac{10}{9}}$ □

4. Let trapezoid $ABCD$ be such that $AB||CD$. Additionally, $AC = AD = 5$, $CD = 6$, and $AB = 3$. Find BC.

   Proposed by Richard Chen

   Solution. $\boxed{4}$

   We see that $BC = \boxed{4}$ since ABC is a $3-4-5$ right triangle. □

5. At Merrick's Ice Cream Parlor, customers can order one of three flavors of ice cream and can have their ice cream in either a cup or a cone. Additionally, customers can choose any combination of the following three toppings: sprinkles, fudge, and cherries. How many ways are there to buy ice cream?

   Proposed by Ezra Erives

   Solution. $\boxed{48}$

   $3 \cdot 2 \cdot 2^3 = \boxed{48}$ □

6. Find the minimum possible value of the expression $|x+1|+|x-4|+|x-6|$.

   Proposed by Taiki Aiba

   Solution. $\boxed{7}$

   We have that $x = 4$ minimizes the expression by inspection. Plugging in $x = 4$ gives us the desired minimum value of the expression, which is 7. □

7. How many 3 digit numbers have an even number of even digits?

   Proposed by Janabel Xia

   Solution. $\boxed{450}$

   Since the last three digits can be any of 0 through 9 (5 even and 5 odd), for any choice of the first digit, half the time the last two digits will have the right parities. Since there are 900 4-digit numbers, we get half, or $\boxed{450}$ as our answer. □

8. Given that the number $1a99b67$ is divisible by 7, 9, and 11, what are a and b? Express your answer as an ordered pair.

   Proposed by Jeff Lin

   Solution. $\boxed{(3,1)}$

   Using the divisibility rules for 9 and 11, we get that $a+b$ leaves a remainder of 4 when divided by 9 and $a-b$ has a remainder of 2 when divided by 11. This leaves $(3,1)$ as our only possibility. We can check that this is divisible by 7 if we want, but it leaves us with an answer of $\boxed{(3,1)}$. □

9. Let O be the center of a quarter circle with radius 1 and $\widehat{AB}$ be the quarter of the circle's circumference. Let M, N be the midpoints of AO and BO, respectively. Let X be the intersection of AN and BM. Find the area of the region enclosed by $\widehat{AB}, AX, BX$.

   Proposed by Sooyoung Choi

   Solution. $\boxed{\frac{\pi}{4}-\frac{1}{3}}$

   We see that X is the centroid of triangle AOB, which we know is a third of height from OA and OB. Let the perpendiculars from X to OA and OB be C and D, respectively. Then $XC = XD = \frac{1}{3}$. We then get that the complementary area is $[XCOD]+[ACX]+[BDX] = \frac{1}{3}^2 + 2\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{2}{3} = \frac{1}{3}$, so the area enclosed by $\widehat{AB}$, BX, and XA is $\boxed{\frac{\pi}{4}-\frac{1}{3}}$. □

10. Each square of a 5-by-1 grid of squares is labeled with a digit between 0 and 9, inclusive, such that the sum of the numbers on any two adjacent squares is divisible by 3. How many such labelings are possible if each digit can be used more than once?

    Proposed by Alex Li

    Solution. $\boxed{1510}$

    We see the possible sequences are (when written as their remainders divided by 3) 00000, 12121, or 21212. Since there are 4 numbers divisible by 3 and 3 numbers that leave a remainder of 1 or 2, the total number of labelings is $4^5 + 2\cdot 3^5 = 1024 + 486 = \boxed{1510}$. □

11. A two-digit number has the property that the difference between the number and the sum of its digits is divisible by the units digit. If the tens digit is 5, how many different possible values of the units digit are there?

    Proposed by Taiki Aiba

    Solution. $\boxed{4}$

    Let the number be $10a+b$, where a and b are the digits of the number. Then, the difference between the number and the sum of its digits is $10a+b-a-b = 9a$. Since we are given that the tens digit is 5, we have that $a = 5$, and the difference is always $9\cdot 5 = 45$. If this difference is divisible by the units digit b, then b must be a factor of 45, but less than 10. We get that b can take on values 1, 3, 5, and 9, so the units digit can take on $\boxed{4}$ different possible values. □

12. There are 2019 red balls and 2019 white balls in a jar. One ball is drawn and replaced with a ball of the other color. The jar is then shaken and one ball is chosen. What is the probability that this ball is red?

Proposed by Ephram Chun

Solution. $\boxed{\frac{1}{2}}$

We can see that if we have n red balls and n white balls than there is an equal chance either of the 2 cases can happen. Case 1 There are $n-1$ white balls and $n+1$ red balls Case 2 There are $n+1$ white balls and $n-1$ red balls

The probability for Case 1 Is $\frac{n+1}{2n}$ The probability for Case 2 is $\frac{n-1}{2n}$ The average of these 2 probabilities will be the answer, therefore the answer is $\frac{1}{2}$. □

13. Let $ABCD$ be a square with side length 2. Let ℓ denote the line perpendicular to diagonal AC through point C, and let E and F be the midpoints of segments BC and CD, respectively. Let lines AE and AF meet ℓ at points X and Y, respectively. Compute the area of $\triangle AXY$.

Proposed by Alex Li

Solution. $\boxed{\frac{8}{3}}$

Let point Z be the foot of the perpendicular from E to AC. Also, note that $EZ = \frac{\sqrt{2}}{2}$, $AZ = \frac{3}{4}AC$, and since $AC = 2\sqrt{2}$, $AZ = \frac{3\sqrt{2}}{2}$. Moreover, since ΔAZE is similar to ΔACX, $XC = \frac{2\sqrt{2}}{3}$. Thus $[AXY] = \left(\frac{2\sqrt{2}}{3}\right)(2\sqrt{2}) = \boxed{\frac{8}{3}}$. □

14. Express $\sqrt{21-6\sqrt{6}}+\sqrt{21+6\sqrt{6}}$ in simplest radical form.

Proposed by Sammy Charney

Solution. $\boxed{6\sqrt{2}}$

Setting this expression equal to c and squaring gives $c^2 = 42+2\sqrt{441-216} = 42+30 = 72$, so the answer is $\sqrt{72} = \boxed{6\sqrt{2}}$. □

15. Let $\triangle ABC$ be an equilateral triangle with side length two. Let D and E be on AB and AC respectively such that $\angle ABE = \angle ACD = 15°$. Find the length of DE.

Proposed by Daniel Hong

Solution. $\boxed{4-2\sqrt{3}}$

By angle chasing, we see that lines BE and CD are perpendicular. Let x be the length of DE. Then, we have $BD = CE = 2-x$. By Pythagoren theorem, we have $BD^2 + CE^2 = DE^2 + BC^2$ since BE and CD is perpendicular. Setting up the equation, we obtain $2^2 + x^2 = (2-x)^2 + (2-x)^2$. Sovling for x gives $x = \boxed{4-2\sqrt{3}}$. □

16. 2018 ants walk on a line that is 1 inch long. At integer time t seconds, the ant with label $1 \le t \le 2018$ enters on the left side of the line and walks among the line at a speed of $\frac{1}{t}$ inches per second, until it reaches the right end and walks off. Determine the number of ants on the line when $t = 2019$ seconds.

Proposed by Kevin Zhao

Solution. $\boxed{1009}$

We see that the ant with index t gets to the edge at $t + \frac{1}{\frac{1}{t}} = 2t$. The ant with index 1009 falls off the end at $t = 2018$, and the ant with index 1010 falls of the end at $t = 2020$, so thus, the first 1009 ants fall off the end. Since every ant either is on the line or fell off the end, there are $2018-1009 = \boxed{1009}$ ants on the line. □

17. Determine the number of ordered tuples $(a_1, a_2, \ldots, a_5)$ of positive integers that satisfy $a_1 \le a_2 \le \cdots \le a_5 \le 5$.

Proposed by Janabel Xia

Solution. $\boxed{126}$

Solution One: We can think of the number of tuples as the number of paths of moves $+(1,0)$ or $+(0,1)$, starting from $(1,0)$ to $(5,5)$. This gives us $\binom{9}{4} = \boxed{126}$ tuples.
Solution Two: Alternatively, the answer is the number of ways to choose the location of nine dividers, corresponding to the increments from $k-1$ to k for $2 \le k \le 5$, amongst ten indistinguishable objects, giving a final answer of $\binom{9}{4} = \boxed{126}$ tuples. □

18. Find the sum of all positive integer values of k for which the equation

$$\gcd(n^2 - n - 2019, n+1) = k$$

has a positive integer solution for n.

Proposed by Jeff Lin

Solution. $\boxed{2018}$

We apply the Euclidean algorithm on this LCM, noting that $n^2 - n - 2$ is divisible by $n+1$, so the LCM is actually $\operatorname{lcm}(2017, n+1)$, which takes the values of 2017 and 1, as 2017 is prime. Therefore, the answer is $\boxed{2018}$. □

19. Let $a_0 = 2$, $b_0 = 1$, and for $n \ge 0$, let

$$a_{n+1} = 2a_n + b_n + 1,$$

$$b_{n+1} = a_n + 2b_n - 1.$$

Find the remainder when a_{2019} is divided by 100.

Proposed by Alex Li

Solution. $\boxed{20}$

Defining $x_n = a_n + b_n$ and $y_n = a_n - b_n$, we see that $x_{n+1} = 3x_n$ and $y_{n+1} = y_n + 2$. Also, since $x_0 = 3$ and $y_0 = 1$, we have $x_{2019} = 3^{2020}$ and $y_{2019} = 2 \cdot 2019 + 1 = 4039$. Thus $a_n = \frac{3^{2020}+4039}{2}$. To find this $\pmod{100}$, we can find the numerator $\pmod{200}$ and divide by 2. Since $\phi(200) = 80$, $3^{2020} + 4039 \equiv 3^{20} + 39 \equiv 43^4 + 39$, where the last result comes from $3^{20} = 243^4$. Since $43^4 \equiv 49^2 \equiv 1$, our desired answer is $\frac{1+39}{2} = \boxed{20}$. □

20. In ΔABC, let AD be the angle bisector of $\angle BAC$ such that D is on segment BC. Let T be the intersection of ray $\overrightarrow{CB}$ and the line tangent to the circumcircle of $\triangle ABC$ at A. Given that $BD = 2$ and $TC = 10$, find the length of AT.

Proposed by Sooyoung Choi

Solution. $\boxed{5+\sqrt{5}}$

First, we claim that $\triangle TAD$ is an isosceles triangle such that $AT = TD$. Since AT is tangent to circumcircle of $\triangle ABC$, we have $\angle TAB = \angle ACD$. Then, by angle chasing we have $\angle ADT = \angle DAC + \angle ACD = \angle BAD + \angle TAB = \angle TAD$. Thus, we have $\angle TAD = \angle TDA$. Let $AT = x$. Then, we have $TB = x - 2$. By power of point, we have $AT^2 = TB \cdot TC$. Then, $x^2 = 10(x-2)$. Solving for x gives us $x = \boxed{5+\sqrt{5}}$ □