Guts Round Solutions

Lexington High School

December 7th, 2019

10th Annual Lexington Math Tournament - Guts Round - Part 1

Team Name: ____________________________

________________ 1. **[5]** A positive integer is said to be *transcendent* if it leaves a remainder of 1 when divided by 2. Find the 1010th smallest positive integer that is transcendent.

Proposed by Ezra Erives

Solution. $\boxed{2019}$

The n-th transcendent number is given by $2n-1$. Plugging in $n = 1010$ yields $\boxed{2019}$. □

________________ 2. **[5]** The two diagonals of a square are drawn, forming four triangles. Determine, in degrees, the sum of the interior angle measures in all four triangles.

Proposed by Alex Li

Solution. $\boxed{720^\circ}$

Each triangle has angles summing up to 180°, so $180^\circ \cdot 4 = \boxed{720^\circ}$ □

________________ 3. **[5]** Janabel multiplied 2 two-digit numbers together and the result was a four digit number. If the thousands digit was nine and hundreds digit was seven, what was the tens digit?

Proposed by Sooyoung Choi

Solution. $\boxed{0}$

The only possible case is $99 \times 98 = 9702$. Thus our answer is $\boxed{0}$. □

10th Annual Lexington Math Tournament - Guts Round - Part 2

Team Name: ____________________________

________________ 4. **[5]** Two friends, Arthur and Brandon, are comparing their ages. Arthur notes that 10 years ago, his age was a third of Brandon's current age. Brandon points out that in 12 years, his age will be double of Arthur's current age. How old is Arthur now?

Proposed by Armaan Tpirneni

Solution. $\boxed{18}$

Let A be Arthur's age, and let B be Brandon's age. Then we are given the two equations $3(A-10) = B$ and $B+12 = 2A$. Solving yields $(A, B) = (18, 24)$, and thus $A = \boxed{24}$. □

______________ 5. **[5]** A farmer makes the observation that gathering his chickens into groups of 2 leaves 1 chicken left over, groups of 3 leaves 2 chickens left over, and groups of 5 leaves 4 chickens left over. Find the smallest possible number of chickens that the farmer could have.

Proposed by Ezra Erives

Solution. $\boxed{29}$

The condition is equivalent to the number of chickens being one less than a multiple of 2, a multiple of 3, and a multiple of 5. The smallest positive integer satisfying these conditions is one less than the least common multiple of these three numbers, and so the answer is $\text{lcm}(2,3,5)-1=\boxed{29}$. □

______________ 6. **[5]** Charles has a bookshelf with 3 layers and 10 indistinguishable books to arrange. If each layer must hold less books than the layer below it and a layer cannot be empty, how many ways are there for Charles to arrange his 10 books?

Proposed by Hannah Shen

Solution. $\boxed{4}$

Let the number on each layer represent how many books are on that layer. For every set of three numbers that add to ten, there is exactly one way to arrange these three numbers on the layers such that the smallest number is on the top, and the largest is on the bottom. (For instance, there is only one way to arrange the numbers 1, 3, and 6 on the shelves - 1 must be on top, then 3, then 6 on the bottom.) The only sets of numbers that satisfy these conditions are $(1,2,7)$, $(1,3,6)$, $(1,4,5)$ and $(2,3,5)$. Thus, the answer is $\boxed{4}$. □

10th Annual Lexington Math Tournament - Guts Round - Part 3

Team Name: ______________________________

______________ 7. **[6]** Determine the number of factors of 2^{2019}.

Proposed by Janabel Xia

Solution. $\boxed{2020}$

Observe that for positive integer n, 2^n has $n+1$ factors. These factors are $2^0, 2^1, dots, 2^n$. Therefore the number of factors of 2^{2019} is $2019+1=\boxed{2020}$ □

______________ 8. **[6]** The points A, B, C, and D lie along a line in that order. It is given that $\overline{AB}:\overline{CD}=1:7$ and $\overline{AC}:\overline{BD}=2:5$. If $BC=3$, find AD.

Proposed by Ezra Erives

Solution. $\boxed{11}$

Let the segments AB, BC, and CD have lengths x, y, and z respectively. Then we have that $\frac{x}{z}=\frac{1}{7}$, $\frac{x+y}{y+z}=25$, and $y=3$. Solving gives $(x,y,z)=(1,3,7)$, and so $x+y+z=\boxed{11}$. □

______________ 9. **[6]** A positive integer n is equal to one-third the sum of the first n positive integers. Find n.

Proposed by Taiki Aiba

Solution. $\boxed{5}$

Using the fact that the sum of the first n positive integers is $\frac{n(n+1)}{2}$, we set up the equation $n=\frac{n(n+1)}{6}$, which simplifies to $n^2-5n=0$. The left-hand side of the equation can be factored as $n(n-5)$. Since n is positive, we can discard $n=0$, which gives us $n=5$ as the only possible value of n. □

10th Annual Lexington Math Tournament - Guts Round - Part 4

Team Name: ________________________________

________________ 10. **[6]** Let the numbers $a, b, c,$ and d be in arithmetic progression. If $a+2b+3c+4d=5$ and $a=\frac{1}{2}$, find $a+b+c+d$.

Proposed by Kevin Zhao

Solution. $\boxed{2}$

Note that $5+20d=5$, $a=b=c=d=\frac{1}{2}$. Thus $a+b+c+d=2$. □

________________ 11. **[6]** Ten people playing brawl stars are split into five duos of 2. Determine the probability that Jeff and Ephram are paired up.

Proposed by Kevin Zhao

Solution. $\boxed{\frac{1}{9}}$

After choosing Jeff, you have 9 ways to choose the remaining person, so there is a $\boxed{\frac{1}{9}}$ chance Jeff and Ephram are on the same team. □

________________ 12. **[6]** Define a sequence recursively by $F_0=0$, $F_1=1$, and for all $n\ge 2$, $F_n=\left\lceil\frac{F_{n-1}+F_{n-2}}{2}\right\rceil+1$, where $\lceil r\rceil$ denotes the least integer greater than or equal to r. Find F_{2019}.

Proposed by Taiki Aiba

Solution. $\boxed{2019}$

I claim that $F_n=n$ for all integers $n\ge 0$. We proceed with strong induction. We see that $F_2=\left\lceil\frac{F_1+F_0}{2}\right\rceil+1=\left\lceil\frac{1}{2}\right\rceil+1=1+1=2$. Now, assume that $F_k=k$ holds for all k from 0 to k. Then, we must show that $F_{k+1}=k+1$. We have that $F_{k+1}=\left\lceil\frac{F_k+F_{k-1}}{2}\right\rceil+1=\left\lceil\frac{k+k-1}{2}\right\rceil+1=\left\lceil\frac{2k-1}{2}\right\rceil+1=k+1$. This completes our strong induction, so $F_n=n$ is true for all integers $n\ge 0$. This means that $F_{2019}=\boxed{2019}$, as desired. □

10th Annual Lexington Math Tournament - Guts Round - Part 5

Team Name: ________________________________

________________ 13. **[7]** Determine the number of different circular bracelets can be made with 7 beads, all either colored red or black.

Proposed by Anka Hu

Solution. $\boxed{18}$

We proceed with casework on the number of red beads. If there are 0 or 1, there is 1 way to make the bracelet. If there are 2, there are 3 ways to make the bracelet. If there are 3, there are 4 ways to make the bracelet. Note that the number of ways to make a bracelet with 4 red beads is equivalent to making a bracelet with 3 black beads, which has the same number of bracelets as one with 4 red beads. Thus there are a total of $2(1+1+3+4)=\boxed{18}$ bracelets. □

________________ 14. **[7]** The product of 260 and n is a perfect square. The 2020th least possible positive integer value of n can be written as $p_1^{e_1}\cdot p_2^{e_2}\cdot p_3^{e_3}\cdot p_4^{e_4}$. Find the sum $p_1+p_2+p_3+p_4+e_1+e_2+e_3+e_4$.

Proposed by Ephram Chun

Solution. $\boxed{131}$

So we can find the prime factorization of 260 which is $2^2 \cdot 5 \cdot 13$ So we know that it γ must be a factor of $5 \cdot 13 = 65$ Therefore the least possible value would be 65.

Then, we find the 2^{nd} lowest number which would be $2^2 \cdot 65$.

The 3^{rd} smallest possible value would be multiplying by $3^2 \cdot 65$

Therefore the 2020^{th} least possible positive integer value of γ would be $2020^2 \cdot 65$.

This can be written as

$$2020^2 \cdot 65 = (2^2 \cdot 5^1 \cdot 101^1)^2 \cdot 65 = 2^4 \cdot 5^2 \cdot 101^2 \cdot 5^1 \cdot 13^1 = 2^4 \cdot 5^3 \cdot 13^1 \cdot 101^2$$

Therefore we just need to find the sum of $2+4+5+3+13+1+101+2$ which is $\boxed{131}$ □

______ 15. **[7]** Let B and C be points along the circumference of circle ω. Let A be the intersection of the tangents at B and C and let $D \neq A$ be on $\overrightarrow{AC}$ such that $AC = CD = 6$. Given $\angle BAC = 60°$, find the distance from point D to the center of ω.

Proposed by Alex Li

Solution. $\boxed{4\sqrt{3}}$

We see that ABD is a 30-60-90 right triangle with $BD = 6\sqrt{3}$. Since D and A are symmetric about the circle, $DO = AO = \boxed{4\sqrt{3}}$ □

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10th Annual Lexington Math Tournament - Guts Round - Part 6

Team Name: ______________________

______ 16. **[7]** Evaluate $\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}$.

Proposed by Sammy Charney

Solution. $\boxed{2}$

Setting this equal to x, we have $x = \sqrt{2+x}$. Solving this yields $x = \boxed{2}$. □

______ 17. **[7]** Let $n(A)$ be the number of elements of set A and $||A||$ be the number of subsets of set A. Given that $||A|| + 2||B|| = 2^{2020}$, find the value of $n(B)$.

Proposed by Janabel Xia

Solution. $\boxed{2018}$

$2||B|| = 2^{2019} \to ||B|| = 2^{2018} \to n(B) = 2018$. □

______ 18. **[7]** a and b are positive integers and $8^a 9^b$ has 578 factors. Find ab.

Proposed by Kevin Zhao

Solution. $\boxed{88}$

We see that $8^a 9^b = 2^{3a} 3^{2b}$, and that has $(3a+1)(2b+1)$ factors. On the other hand, we see that $578 = 2 \cdot 17^2$. Let $3a+1 = m$ and $2b+1 = n$, so $mn = 2 \cdot 17^2$. Doing $\mod 2$, we see that n must be odd.

We see, that there are no solutions which are $m = 2$ or $n = 2$, since plugging in, we get non-integers. Thus, we see that our other solutions are $34 \cdot 17$ and $578 \cdot 1$. Since a and b are positive, then neither m nor n can be 1. Now, thus, since m is odd, then we have $m = 34$ and $n = 17$ so $a = 11$ and $b = 8$. As a result, $ab = (11)(8) = \boxed{88}$ □

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10th Annual Lexington Math Tournament - Guts Round - Part 7

Team Name: ______________________________

______________ 19. **[8]** Determine the probability that a randomly chosen positive integer is relatively prime to 2019.

Proposed by Sammy Charney

Solution. $\boxed{\dfrac{448}{673}}$

By the Principle of Inclusion and Exclusion, there are $2019-673-3+1=1344$ numbers relatively prime to 2019 in a given set of 2019 consecutive integers. The answer is $\frac{1344}{2019}=\boxed{\dfrac{448}{673}}$. ☐

______________ 20. **[8]** A 3-by-3 grid of squares is to be numbered with the digits 1 through 9 such that each number is used once and no two even-numbered squares are adjacent. Determine the number of ways to number the grid.

Proposed by Alex Li

Solution. $\boxed{17280}$

There are 4 even-numbered squares. If we color the board like a checker-board such that there are 4 black squares and 5 white squares, we see that the even numbers must either be all on black squares or all on white squares. If they are all on white squares, there are 5 ways to choose the white square with an odd number. Thus the total number of colorings is $5!\cdot 4!+5\cdot 5!\cdot 4!=\boxed{17280}$. ☐

______________ 21. **[8]** In ΔABC, point D is on AC so that $\frac{AD}{DC}=\frac{1}{13}$. Let point E be on BC, and let F be the intersection of AE and BD. If $\frac{DF}{FB}=\frac{2}{7}$ and the area of ΔDBC is 26, compute the area of ΔFAB.

Proposed by Ephram Chun

Solution. $\boxed{\dfrac{14}{9}}$

We have that $\frac{AD}{DC}=\frac{1}{13}$, so by area ratios, we have that the ratio of the areas of ABD and DBC is also $\frac{1}{13}$. Given that the area of DBC is 26, the area of ABD is $\frac{1}{13}\cdot 26=2$.

We have that $\frac{DF}{FB}=\frac{2}{7}$, so by area ratios, we have that the ratio of the areas of DAF and FAB is also $\frac{2}{7}$. Then, we have that the ratio of the areas of FAB and ABD is $\frac{7}{2+7}=\frac{7}{9}$. Given that the area of ABD is 2, we have that the area of FAB is $\frac{7}{9}\cdot 2=\boxed{\dfrac{14}{9}}$. ☐

10th Annual Lexington Math Tournament - Guts Round - Part 8

Team Name: ______________________________

______________ 22. **[8]** A quarter circle with radius 1 is located on a line with its horizontal base on the line and to the left of the vertical side. It is then rolled to the right until it reaches its original orientation. Determine the distance traveled by the center of the quarter circle.

Proposed by Sammy Charney

Solution. $\boxed{\dfrac{3\pi}{2}}$

The point traces out the following paths:

1. A quarter circle with radius 1, which comes from pivoting around one end of the radius.
2. A straight line with length equal to that of a quarter circle with radius 1, which comes from rolling along the circumference of the quarter circle.
3. A final quarter circle with radius 1, which comes from pivoting around the other end of the radius.

Each step has the center traveling $\frac{\pi}{2}$, meaning the center travels a total of $\boxed{\frac{3\pi}{2}}$. □

________ 23. **[8]** In 1734, mathematician Leonhard Euler proved that $\frac{\pi^2}{6} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$. With this in mind, calculate the value of $\frac{1}{1} - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \dots$ (the series obtained by negating every other term of the original series).

Proposed by Ezra Erives

Solution. $\boxed{\frac{\pi^2}{12}}$

Let $S = \frac{\pi^2}{6}$ be the original series. Then the desired series equals $S - 2(\frac{S}{4}) = \frac{S}{2} = \boxed{\frac{\pi^2}{12}}$. □

________ 24. **[8]** Billy the biker is competing in a bike show where he can do a variety of tricks. He knows that one trick is worth 2 points, 1 trick is worth 3 points, and 1 is worth 5 points, but he doesn't remember which trick is worth what amount. When it's Billy's turn to perform, he does 6 tricks, randomly choosing which trick to do. Compute the sum of all the possible values of points that Billy could receive in total.

Proposed by Sammy Charney

Solution. $\boxed{370}$

He can make all numbers from 12 to 30 except for 29. Therefore, the answer is $\boxed{370}$ □

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10th Annual Lexington Math Tournament - Guts Round - Part 9

Team Name: ____________________

________ 25. **[9]** Find the largest prime factor of 1031301.

Proposed by Jeff Lin

Solution. $\boxed{163}$

We approach by trying to prime factorize this number. We can write this number as $101^3 + 10^3$. Using the sum of cubes formula, we can write this as $(111)(101^2 - 101 * 10 + 10^2)$. Expanding the write part gives us $(3)(37)(10201 - 1010 - 100) = (3)(37)(9291)$. We now have to factor 9291. This is clearly divisible by 3, written as $3 * 3097$. Trying small primes, we see that $3097 = 19 * 163$. As 163 is prime and bigger than 37, our answer is $\boxed{163}$. □

________ 26. **[9]** Let $ABCD$ be a trapezoid such that $AB \parallel CD, \angle ABC = 90°, AB = 5, BC = 20, CD = 15$. Let X, Y be the intersection of the circle with diameter BC and segment AD. Find the length of XY.

Proposed by Sooyoung Choi

Solution. $\boxed{4\sqrt{5}}$

We notice that one of our intersection is the midpoint of AD since $\frac{5+15}{2} = 10$. Also, we see that the length of $AD = 10\sqrt{5}$ due to the pythagorean theorem. Let X be the midpoint intersection and Y be the another intersection. Using Power of Point, we have $AB^2 = AX \cdot AY$. Then, $5^2 = AY \cdot 5\sqrt{5}$. Thus, $AY = \sqrt{5}$, and the length of XY is equal to $5\sqrt{5} - \sqrt{5} = \boxed{4\sqrt{5}}$ □

______________ 27. **[9]** A string consisting of 1's, 2's, and 3's is said to be a superpermutation of the string 123 if it contains every permutation of 123 as a contiguous substring. Find the smallest possible length of such a superpermutation.

Proposed by Ezra Erives

Solution. $\boxed{9}$

One example is 123121321. □

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10th Annual Lexington Math Tournament - Guts Round - Part 10

Team Name: ______________________________

______________ 28. **[11]** Suppose that we have a function $f(x) = x^3 - 3x^2 + 3x$, and for all $n \geq 1$, $f^n(x)$ is defined by the function f applied n times to x. Find the remainder when $f^5(2019)$ is divided by 100.

Proposed by Ezra Erives

Solution. $\boxed{33}$

Observe that $f(x)$ may be rewritten as $(x-1)^3+1$, from which it follows that $f^n(x) = (x-1)^{3^n}+1$. Thus, $f^5(2019) = 2018^{243}+1 \equiv \boxed{33}$. □

______________ 29. **[11]** A function $f : \{1,2,\ldots,10\} \to \{1,2,\ldots,10\}$ is said to be *happy* if it is a bijection and for all $n \in \{1,2,\ldots,10\}$, $|n - f(n)| \leq 1$. Compute the number of happy functions.

Proposed by Ezra Erives

Solution. $\boxed{89}$

Let $T(n)$ be the number of n-happy functions (when 10 is replaced generic n). We claim that $T(n) = F_{n+1}$, where F_k is the k-th Fibonacci number. For $n > 1$, let f_n be an n-happy function. If $f_n(n) = n$, then there are $T(n-1)$ ways to construct f_n. If $f_n(n) = n-1$, then there are $T(n-2)$ ways to construct f_n. Thus, $T(n) = T(n-1) + T(n-2)$. The claim then follows from the fact that $f(1) = 1$ and $f(2) = 2$. Finally, $T(10) = F_{11} = \boxed{89}$. □

______________ 30. **[11]** Let ΔLMN have side lengths $LM = 15$, $MN = 14$, and $NL = 13$. Let the angle bisector of $\angle MLN$ meet the circumcircle of ΔLMN at a point $T \neq L$. Determine the area of ΔLMT.

Proposed by Kevin Zhao

Solution. $\boxed{60}$

First things first: We see that if we draw T's altitude to MN and let the intersection be H, which is a perpendicular bisector since $\angle MLT = \angle NLT$, meaning $MT = NT$, and then using SSA with right angles.

Using a Heron mini-bash shows that the area is 84.

Then, we find the circumradius: $\frac{13\cdot 14\cdot 15}{4\cdot 84} = \frac{65}{8}$. Using Pythagoras, we see that $HT = \frac{65}{8} - \sqrt{(\frac{65}{8})^2 - 7^2} = 4$.

Let the intersection of LT and MN be G. We see, from similarity, that $MG = 7.5$.

The sum of areas of LMG and GMT is the area of LMT, so we use the typical base-height formula: $\frac{7.5(4+12)}{2} = \boxed{60}$. □

10th Annual Lexington Math Tournament - Guts Round - Part 11

Team Name: ______________________________

______________ 31. **[13]** Find the value of

$$\sum_{d|2200} \tau(d),$$

where $\tau(n)$ denotes the number of divisors of n, and where $a|b$ means that $\frac{b}{a}$ is a positive integer.

Proposed by Richard Chen

Solution. $\boxed{180}$

Let

$$T(n) = \sum_{d|n} \tau(d).$$

The crucial observation is that T is multiplicative. That is, for $n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$, $T(n) = T(p_1^{e_1})T(p_2^{e_2})\cdots T(p_k^{e_k})$. In other words, we may solve the problem for each prime factor, and then combine our results. Notice that since $2200 = 2^3 \cdot 5^2 \cdot 11^1$, and so the desired quantity can be expressed by $T(2200) = T(2^3)T(5^2)T(11^1)$. Since

$$T(2^3) = \tau(1) + \tau(2) + \tau(4) + \tau(8) = 10$$
$$T(5^2) = \tau(1) + \tau(5) + \tau(25) = 6$$
$$T(11^1) = \tau(1) + \tau(11) = 3$$

we find that the answer is $10 \cdot 6 \cdot 3 = \boxed{180}$. □

________________ 32. **[13]** Let complex numbers $\omega_1, \omega_2, \dots, \omega_{2019}$ be the solutions to the equation $x^{2019} - 1 = 0$. Evaluate

$$\sum_{i=1}^{2019} \frac{1}{1+\omega_i}$$

Proposed by Alex Li

Solution. $\boxed{\frac{2019}{2}}$

$\frac{1}{1+\omega_i}$ are the solutions to the equation $\left(\frac{1}{x}-1\right)^{2019} = 1$, which expands out to $2x^{2019} - 2019x^{2018} + \cdots = 0$. By Vieta's, the sum of the solutions is $\boxed{\frac{2019}{2}}$. □

________________ 33. **[13]** Let M be a nonnegative real number such that $x^{x^{x^{\cdots}}}$ diverges for all $x > M$, and $x^{x^{x^{\cdots}}}$ converges for all $0 < x \le M$. Find M.

Proposed by Jeff Lin

Solution. $\boxed{e^{\frac{1}{e}}}$

I claim that the answer is $e^{\frac{1}{e}}$. To prove this, let $x^{x^{x^{x}}} = y$. Then, $x^y = y$, so $x = y^{\frac{1}{y}}$. Therefore, our problem boils down to finding the maximum value of $y^{\frac{1}{y}}$. We have $e^n \ge 1 + n$ for all n, by definition of e. Letting n equal $\frac{n-e}{e}$, we have $e^{\frac{n}{e}-1} \ge 1 + \frac{n-e}{e}$, so $\frac{e^{\frac{n}{e}}}{e} \ge \frac{n}{e}$, and $e^{\frac{1}{e}} \ge n^{\frac{1}{n}}$ for all n, as desired. □

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10th Annual Lexington Math Tournament - Guts Round - Part 12

Team Name: ______________________________

______________ 34. **[15]** Estimate the number of digits in $\binom{2019}{1009}$. If your estimate is E and the actual value is A, your score for this problem will be

$$\max\left(0, \left\lfloor 15-10\cdot\left|\log_{10}\left(\frac{A}{E}\right)\right|\right\rfloor\right).$$

Proposed by Janabel Xia

Solution. $\boxed{607}$ ☐

______________ 35. **[15]** You may submit any integer E from 1 to 30. Out of the teams that submit this problem, your score will be

$$\frac{E}{2(\text{the number of teams who chose } E)}.$$

Proposed by Janabel Xia

Solution. $\boxed{None}$ ☐

______________ 36. **[15]** We call a $m\times n$ *domino-tiling* a configuration of 2×1 dominoes on an $m\times n$ cell grid such that each domino occupies exactly 2 cells of the grid and all cells of the grid are covered. How many 8×8 domino-tilings are there? If your estimate is E and the actual value is A, your score for this problem will be

$$\max\left(0, \left\lfloor 15-10\cdot\left|\log_{10}\left(\frac{A}{E}\right)\right|\right\rfloor\right).$$

Proposed by Janabel Xia

Solution. $\boxed{12988816}$

Article linked here: http://math.uchicago.edu/ may/REU2015/REUPapers/Borys.pdf ☐