Individual Round Solutions

Lexington High School

April 7, 2018

1. Evaluate $6^4+5^4+3^4+2^4$.

   Proposed by Evan Fang

   Solution. $\boxed{2018}$

   Use Addition $\square$

2. What digit is most frequent between 1 and 1000 inclusive?

   Proposed by 1

   Solution. $\boxed{1}$

   1 through 9 appear the same number of times for one-digit, two-digit, and three-digit numbers by symmetry. But we're also including $1,000$ so 1 appears the most. $\square$

3. Let $n=\gcd(2^2\cdot 3^3\cdot 4^4, 2^4\cdot 3^3\cdot 4^2)$. Find the number of positive integer factors of n.

   Proposed by Evan Fang

   Solution. $\boxed{36}$

   The GCD is $2^8\cdot 3^3$ which has $9\cdot 4=36$ factors. $\square$

4. Suppose p and q are prime numbers such that $13p+5q=91$. Find $p+q$.

   Proposed by Evan Fang

   Solution. $\boxed{15}$

   Notice that 91 is a multiple of 13 thus $Q=13$ from which we find out that $P=2$. $\square$

5. Let $x=(5^3-5)(4^3-4)(3^3-3)(2^3-2)(1^3-1)$. Evaluate 2018^x.

   Proposed by Evan Fang

   Solution. $\boxed{1}$

   $x=0\to 2018^0=1$ $\square$

6. Liszt the lister lists all 24 four-digit integers that contain each of the digits $1,2,3,4$ exactly once in increasing order. What is the sum of the 20th and 18th numbers on Liszt's list?

   Proposed by Nathan Ramesh

   Solution. $\boxed{7553}$

   The relevant numbers are 4132 and 3412, respectively. Their sum is 7553. $\square$

7. Square $ABCD$ has center O. Suppose M is the midpoint of AB and $OM+1=OA$. Find the area of square $ABCD$.

   Proposed by Evan Fang

Solution. $\boxed{12+8\sqrt{2}}$

We have $OM+1 = OM\sqrt{2}$ and thus $OM = \sqrt{2}+1 \to AB = 2\sqrt{2}+2$. We get the answer by squaring AB. □

8. How many positive 4-digit integers have at most 3 distinct digits?

Proposed by Evan Fang

Solution. $\boxed{4464}$

There are 9000 4-digit numbers and $9\cdot 9\cdot 8\cdot 7 = 4536$ 4-digit numbers with 4 distinct digits. Thus the answer is $9000-45356 = 4464$. □

9. Find the sum of all distinct integers obtained by placing + and − signs in the following spaces

$$2\_3\_4\_5.$$

Proposed by Evan Fang

Solution. $\boxed{16}$

Each number you obtain is distinct. Then notice that if you pair every combination up with it's "conjugate" you will always obtain a sum of 4. For example $(2+3+4-5)+(2-3-4+5) = 4$. Thus, the answer is just $\frac{2^3}{2}\cdot 4 = 16$. □

10. In triangle $ABC, \angle A = 2\angle B$. Let I be the intersection of the angle bisectors of B and C. Given that $AB = 12, BC = 14$, and $CA = 9$, find AI.

Proposed by Evan Fang

Solution. $\boxed{\frac{24}{5}}$

Since I is the incenter, AI is the angle bisector of angle A. Let AI hit BC at K. Then by angle bisector theorem $BK = 8$. But since $\angle KAB = \angle AKB$ we have $AK = 8$ as well. Then, applying angle bisector theorem again on triangle ABI we have $AI = \frac{3}{5}\cdot 8 = \frac{24}{5}$. □

11. You have a $3\times 3\times 3$ cube in front of you. You are given a knife to cut the cube and you are allowed to move the pieces after each cut before cutting it again. What is the minimum number of cuts you need to make in order to cut the cube into 27 $1\times 1\times 1$ cubes?

Proposed by Sooyoung Choi

Solution. $\boxed{6}$

Consider the $1\times 1\times 1$ cube in the middle. In order to get this, you need to cut at least 6 times. Actual case for 6 cut is easy so our answer is 6. □

12. How many ways can you choose 3 distinct numbers from the set $\{1,2,3,\ldots,20\}$ to create a geometric sequence?

Proposed by Evan Fang

Solution. $\boxed{10}$

Clearly, the ratio must be rational. Thus, let the ratio be $\frac{a}{b}$. Thus, if the smallest number is x then we must have $x, \frac{ax}{b}, \frac{a^2x}{b^2}$. Since $\frac{a}{b}$ is minimal, we must have x not be squarefree or else $b = 1$. For $b = 1$, we see that the ratio can be 3 for $x = 1,2$ and 2 for $x = 1,2,3,4,5$.

Note that only $4,8,9,12,16,18$. For those divisible by 4, we simply need to check $\frac{3}{2}$, which works for $x = 4,8$ and $\frac{5}{2}$ which works for none of them so bigger ones can't work. For those divisble by 9, we check $\frac{4}{3}$ which works for $x = 9$ and $x = \frac{5}{3}$ which fails to work for any of them.

Thus we have a grand total of $2+5+2+1 = 10$ 3-term geometric sequences □

13. Find the sum of all multiples of 12 that are less than 10^4 and contain only 0 and 4 as digits.

Proposed by Evan Fang

Solution. $\boxed{13332}$

A number is a multiple of 12 if and only if it's also a multiple of 3 and 4. Since it's a multiple of 3, we must have 3 4s within the expression. No matter what combination of 0 and 4 we use, the last 2 digits will always be a multiple of 4. Thus, the only possibilities are $4440, 4404, 4044, 444$ giving us a sum $13,332$. □

14. What is the smallest positive integer that has a different number of digits in each base from 2 to 5?

Proposed by Peter Rowley

Solution. $\boxed{81}$

The answer is 81. One approach is to just straight bash the numbers. Suppose the desired number is n. The key observation to the bash is that if n has x digits base p then we must have $p^{x-1} \le n < p^x$.

Say n has 4 digits base 2 which forces 1 digit base 5 which you can easily verify to not work. Similarly, if there are 5 digits base 2 then it is easy to see there must be 2 digits base 5 and 4 digits base 3 which doesn't work as this implies $5 \le n \le 24, 27 \le n \le 80$. Keep doing this until you find something that works.

Alternatively, one can have intuition that n won't have x digits base 2, $x-1$ digits base 3 because we know that $x \ge 4$ and that $2^{x-1} < 3^{x-2}$ for $x \ge 4$. Thus, we guess that n has x digits base 2, $x-2$ digits base 3, $x-3$ base 4 and $x-4$ base 5. One can easily check that $x = 7$ is the first one that works and the rest is easy. □

15. Given 3 real numbers (a, b, c) such that

$$\frac{a}{b+c} = \frac{b}{3a+3c} = \frac{c}{a+3b},$$

find all possible values of

$$\frac{a+b}{c}.$$

Proposed by Evan Fang

Solution. $\boxed{\dfrac{22}{13}}$

By all the equivalences we get $\frac{a}{b+c} = \frac{b}{3a+3c} = \frac{c}{a+3b} = \frac{a+b+c}{4a+4b+4c} = \frac{1}{4}$. Then, some bashing reveals $13b = 15c$ and $13a = 7c$ and hence $a + b = \frac{22}{13}c$ as desired. □

16. Let S be the set of lattice points (x, y, z) in $\mathbb{R}^3$ satisfying $0 \le x, y, z \le 2$. How many distinct triangles exist with all three vertices in S?

Proposed by Yiming Zheng

Solution. $\boxed{2876}$

There are $\binom{27}{3}$ ways to choose three points. However we need to subtract when these points are collinear:

i) Edges: there are 12 of them. ii) Other triples that are collinear on a face: there are 4 per face, giving 24 total. iii) They are collinear and containing the center point $(1,1,1)$: there are 13 of these, one for each pair of points when the center is excluded.

Thus the requested answer is $\binom{27}{3} - 12 - 24 - 13 = 2876$ □

17. Let $\oplus$ be an operator such that for any 2 real numbers a and b, $a \oplus b = 20ab - 4a - 4b + 1$. Evaluate

$$\frac{1}{10} \oplus \frac{1}{9} \oplus \frac{1}{8} \oplus \frac{1}{7} \oplus \frac{1}{6} \oplus \frac{1}{5} \oplus \frac{1}{4} \oplus \frac{1}{3} \oplus \frac{1}{2} \oplus 1.$$

Proposed by Evan Fang

Solution. $\boxed{\dfrac{1}{5}}$

Notice that $\frac{1}{5} \oplus a = \frac{1}{5}$ and $a \oplus \frac{1}{5} = \frac{1}{5}$. The conclusion follows. □

18. A function $f : \mathbb{N} \to \mathbb{N}$ satisfies $f(f(x)) = x$ and $f(2f(2x+16)) = f\left(\frac{1}{x+8}\right)$ for all positive integers x. Find $f(2018)$.

    Proposed by Daniel Liu

Solution. $\boxed{\dfrac{1}{2042}}$

$$f(2f(2x+16)) = f(\frac{1}{x+8}) \implies f(f(2f(2x+16))) = f(f(\frac{1}{x+8}))$$

$$\implies 2f(2x+16) = \frac{1}{x+8}$$

$$\implies f(2018) = \frac{1}{2018}$$

□

19. There exists an integer divisor d of 240100490001 such that $490000 < d < 491000$. Find d.

    Proposed by Euhan Kim

Solution. $\boxed{490701}$

We see that $2401 = 7^4$ and that $49 = 7^2$ and that those strings have 8 and 4 digits, respectively. This suggests that the number could be factored in terms of $x = 490000$. The factorization we find from this is $x^2 + x + 1$, which cannot be factored. However, we notice x is square and can be square rooted to make an expression in terms of $y = \sqrt{x} = 700$: $y^4 + y^2 + 1$. This can be factored to get $(y^2 + y + 1)(y^2 - y + 1) = 490701 * 489301$. Since 490701 is the only one of the two between 490000 and 491000, it is our answer. □

20. Let a and b be not necessarily distinct positive integers chosen independently and uniformly at random from the set $\{1, 2, 3, \cdots, 511, 512\}$. Let $x = \frac{a}{b}$. Find the probability that $(-1)^x$ is a real number.

    Proposed by Evan Fang

Solution. $\boxed{\dfrac{2^{19}+1}{3 \cdot 2^{18}}}$

Note that $(-1)^x$ is real if and only if the denominator of the simplified fraction $\frac{a}{b}$ is odd. This occurs when $v_2(a) \geq v_2(b)$, where $v_2(n)$ denotes the exponent of the largest power of 2 that divides n.

Observe that the number of pairs (a, b) that satisfy $v_2(a) > v_2(b)$ is equal to the number of pairs (a, b) that satisfy $v_2(a) < v_2(b)$. So it remains to deal with the case when $v_2(a) = v_2(b)$. However, it is easy to see that there are 2^8 values of a such that $v_2(a) = 0$, 2^7 values of a such that $v_2(a) = 1, \ldots$, 1 value of a such that $v_2(a) = 8$, and 1 value of a for $v_2(a) = 9$. Therefore, the number of pairs (a, b) for which $v_2(a) = v_2(b)$ is:

$$2^{16} + 2^{14} + \cdots + 2^4 + 1 + 1 = \frac{2^{18} - 1}{3} + 1 = \frac{2^{18} + 2}{3}$$

Since there are 2^{18} possibilities in total, the probability that a randomly chosen pair (a, b) satisfies $v_2(a) = v_2(b)$ is $\frac{2^{18}+2}{3 \cdot 2^{18}}$. Call this probability p.

The probability we want is

$$p + \frac{1-p}{2} = \frac{1+p}{2} = \frac{4 \cdot 2^{18} + 2}{3 \cdot 2^{19}} = \frac{2^{19}+1}{3 \cdot 2^{18}}$$

□

21. In $\triangle ABC$ we have $AB = 4, BC = 6$, and $\angle ABC = 135°$. $\angle ABC$ is trisected by rays B_1 and B_2. Ray B_1 intersects side CA at point F, and ray B_2 intersects side CA at point G. What is the area of $\triangle BFG$?

    Proposed by Richard Chen

    Solution. $\boxed{\dfrac{468-324\sqrt{2}}{7}}$

    Coordinate bashing by setting B as $(0,0)$ and C as the point $(6,0)$ gives B_1 as the y-axis, B_2 as the line $y = x$, and line AC as $y = -\frac{\sqrt{2}}{\sqrt{2}+3}x + \frac{6\sqrt{2}}{\sqrt{2}+3}$, which immediately gives $F = (0, \frac{6\sqrt{2}}{\sqrt{2}+3})$. To find G, we find the intersection of AC with the line $y = x$, which is at $x = \frac{6\sqrt{2}}{2\sqrt{2}+3}$. Then the area of $\triangle BFG = \frac{1}{2}\cdot\frac{6\sqrt{2}}{\sqrt{2}+3}\cdot\frac{6\sqrt{2}}{2\sqrt{2}+3} = \frac{468-324\sqrt{2}}{7}$ □

22. A level number is a number which can be expressed as $x\cdot\lfloor x\rfloor\cdot\lceil x\rceil$ where x is a real number. Find the number of positive integers less than or equal to 1000 which are also level numbers.

    Proposed by Evan Fang

    Solution. $\boxed{331}$

    Setting x to be an integer gives that all perfect cubes are level numbers. Else let $x = p + \frac{r}{q}$ with $0 < \frac{r}{q} < 1$, $q > 1$, $p > 0$, and q, r coprime. For our purposes, we may restrict $p \le 9$. We have $x\cdot\lfloor x\rfloor\cdot\lceil x\rceil = p(p+1)(p+\frac{r}{q})$. For x to induce a level number we need for $q|r(p)(p+1) \implies q|p(p+1)$. The number of r for a fixed $q > 1$ is $\varphi(q)$. Thus, the number of triples (p,q,r) is

    $$\sum_{p=1}^{9}\sum_{\substack{q|p(p+1)\\ q\neq 1}}\varphi(q) = \sum_{p=1}^{9} p(p+1) - 1 = 321$$

    by a well-known identity. It is easy to check that none of these recount perfect cubes or themselves. In total, there are $321 + 10 = 331$ level positive integers at most 1000.

    Solution 2: A level number is a number which can be expressed as $x\cdot\lfloor x\rfloor\cdot\lceil x\rceil$ where x is a real number. Find the number of positive integers less than or equal to 1000 which are also level numbers. If x is an integer this expression becomes x^3, so all cubes are level numbers. Otherwise, assume x is not an integer.

    If $n-1 < x < n$ then this expression is $(n-1)\cdot n\cdot x$. Thus, if this expression is integer then the $x = n-1+\frac{p}{n(n-1)}$ where $0 < p < n(n-1)$ and p is integer.

    Thus, we have $(n-1)\cdot n\cdot(n-1+\frac{p}{n(n-1)}) = n(n-1)^2 + p$. This means that an integer is level if and only if it can be expressed as $n(n-1)^2 + p$ where n, p are integers and $0 < p < n(n-1)$

    Clearly, we must have $2 \le n \le 10$. Thus, the answer is just $\sum_{n=2}^{10} n(n-1) - 1 = 321$

    The final answer is $321 + 10 = 331$ □

23. Triangle $\triangle ABC$ has sidelengths $AB = 13, BC = 14, CA = 15$ and circumcenter O. Let D be the intersection of AO and BC. Compute BD/DC.

    Proposed by Nathan Ramesh, Ezra Erives

    Solution. $\boxed{\dfrac{169}{125}}$

    Let E be the point on BC such that AD and AE are isogonal. It is well known (and easy to show with the law of sines) that

    $$\frac{BD}{CD}\cdot\frac{BE}{CE} = \frac{AB^2}{AC^2}.$$

    Further, since O and H are isogonal conjugates, E is the projection of A onto BC. It easily follows that $\frac{BE}{CE} = \frac{5}{9}$. This gives

    $$\frac{BD}{CD} = \frac{AB^2}{AC^2}\cdot\frac{CE}{BE} = \frac{169}{225}\cdot\frac{9}{5} = \frac{169}{125}.$$

    □

24. Let $f(x) = x^4 - 3x^3 + 2x^2 + 5x - 4$ be a quartic polynomial with roots a, b, c, d. Compute
$$\left(a+1+\frac{1}{a}\right)\left(b+1+\frac{1}{b}\right)\left(c+1+\frac{1}{c}\right)\left(d+1+\frac{1}{d}\right).$$

Proposed by Nathan Ramesh

Solution. $\boxed{-\frac{133}{4}}$

Note that
$$a+1+\frac{1}{a} = \frac{a^2+a+1}{a} = \frac{(\omega-a)(\omega^2-a)}{a},$$
where $\omega = -\frac{1}{2} + i \cdot \frac{\sqrt{3}}{2}$. Thus, the expression is equal to
$$\begin{aligned}\frac{f(\omega)\cdot f(\omega^2)}{f(0)} &= \frac{(-11+2\sqrt{3}\cdot i)(-11-2\sqrt{3}\cdot i)}{-4}\\ &= -\frac{133}{4}.\end{aligned}$$

□

25. Triangle $\triangle ABC$ has centroid G and circumcenter O. Let D be the foot of the altitude from A to BC. If $AD = 2018$, $BD = 20$, and $CD = 18$, find the area of triangle $\triangle DOG$.

Proposed by Nathan Ramesh

Solution. $\boxed{\frac{30}{1009}}$

Let OG intersect AD at H. Since H lies on AD and OG, the existence of the Euler line implies that H is in fact the the orthocenter of $\triangle ABC$. Then we compute
$$[DOG] = \frac{1}{2}\cdot[DGH] = \frac{1}{4}\cdot DH \cdot d(G, AD),$$
where the first equality follows from Euler line ratios. Solving for $d(G, AD) = \frac{|BD-CD|}{3}$ is trivial by coordinates, so it remains to solve for DH nicely. Reflect A over BC to A'. It is trivial by angle chasing that A' lies on the circumcircle of $\triangle BHC$. Then, power of a point gives $A'D \cdot DH = BD \cdot CD$, which implies $DH = \frac{BD\cdot CD}{AD}$. The answer is
$$\frac{BD\cdot CD\cdot|BD-CD|}{12\cdot AD} = \frac{20\cdot 18\cdot 2}{12\cdot 2018} = \frac{30}{1009}.$$

□