Theme Round Solutions

Lexington High School

April 8, 2017

Building

SOLUTION: If John takes 60 hours to build the bridge, he builds $\frac{1}{60}$ of the bridge in an hour. So peter builds $\frac{1}{45}$ of the bridge in an hour. Working together, they build $\frac{1}{60}+\frac{1}{45}=\frac{7}{180}$ of the bridge in an hour. So it would take $\frac{180}{7}$ hours to build the bridge. Bob wants to build a bridge. This bridge has to be an arch bridge which reaches down on each side 10 feet and crosses a 20 foot gap. If the arch is shown by a semi circle with radius 9, and the bridge is 5 feet wide, how many cubic feet of material does Bob need? *Proposed by: Ben Defay* **ANSWER:** $\boxed{1000-\frac{405\pi}{2}}$ **SOLUTION:** Imagine taking a rectangular prism with dimensions 10 ft high, 20 ft long, and 5 ft wide. The material needed is equivalent to what is left if the semicircle with width 5 ft and radius 9 ft is removed from the prism. The prism is $10\cdot 20\cdot 5=1000$ cubic feet. The material removed is $\frac{1}{2}\cdot\pi\cdot 9^2\cdot 5=\frac{405\pi}{2}$. So he would need $1000-\frac{405\pi}{2}$ cubic feet of material. Evan constructs a "poly-chain" by connecting regular polygons of side length 1 and having each adjacent polygon share a side. Additionally, Evan only creates a "poly-chain" if the polygons in the chain all have a consecutive number of side lengths. For each "poly-chain", Evan then assigns it an ordered pair (m,n), where m is the number of polygons in the "poly-chain", and n is the number of sides of the largest polygon. Find all ordered pairs (m,n) that correspond to a "poly-chain" with perimeter 17. *Proposed by: John Guo* **ANSWER:** $\boxed{(1,17),(2,10),(3,8),(5,7)}$ **SOLUTION:** The perimeter of a "poly-chain" is equal to the (sum of the number of sides of the individual polygons) − (the number of shared sides) ·2. With a small number like 17, we can work case by case to find all such "poly-chains" Case 0: 1 Polygon → Trivially, this case has one polygon with 17 sides, giving us $(1,17)$. Case 1: 2 Polygons → With 2 polygons, we are looking for their side lengths to sum up to $17+2\cdot 1=19$. These polygons have length 9 and 10 which gives us $(2,10)$. Case 2: 3 Polygons → With 3 polygons, we are looking for their side lengths to sum up to $17+2\cdot 2=21$. These polygons have side length 6, 7, and 8, which gives us $(3,8)$. Case 3: 4 Polygons → With 4 polygons, we are looking for their side lengths to sum up to $17+2\cdot 3=23$. However, 4 consecutive numbers cannot sum to 23, so there is no solution for 4 polygons. Case 4: 5 Polygons → With 5 polygons, we are looking for their side lengths to sum up to $17+2\cdot 4=25$. These polygons have lengths 3, 4, 5, 6, and 7, giving us $(5,7)$. Since the smallest number of side lengths a regular polygon can have is 3, we know this is the final case. Thus, all possible ordered pairs are $(1,19),(2,10),(3,8),(5,7)$ Ben constructs a triangle ABC such that if M is the midpoint of BC then $AM=BC=10$. Find the sum of all possible integer valued perimters of $\triangle ABC$. *Proposed by: Evan Fang* **ANSWER:** $\boxed{32}$ **SOLUTION:** Let $AB=c$ and $AC=b$. Now, apply stewarts theorem to get that $5\cdot 5\cdot 10+10\cdot 10\cdot 10=5\cdot c\cdot c+5\cdot b+\cdot b \implies 250=b^2+c^2$. In order to have an integer perimeter, we must have $c=x-b$ for some positive integer x. Thus we have $x^2-2bx+2b^2=250$. Rewrite this as a quadratic in b and solve the quadratic equation $2b^2-(2x)b+(x^2-250)=0 \implies b=\frac{2x\pm\sqrt{4x^2-8x^2+2000}}{4} \implies b=\frac{x\pm\sqrt{500-x^2}}{2}$. So if $b=\frac{x+\sqrt{500-x^2}}{2}$ then $c=\frac{x-\sqrt{500-x^2}}{2}$ and vice versa. Now we check for when $x-\sqrt{500-x^2}>0$. This is just $x^2>500-x^2 \iff x^2>250 \iff x>15$ for integers x. And we also have $x^2<500 \iff x<23$ for integers x. Next, we check that each value for x satisfies the triangle inequality. Since $b+c=x>10$ for each x, this side is satisfied. Now we only need to check that $10+x-\sqrt{500-x^2}>x+\sqrt{500-x^2}$ but this is $10>2\sqrt{500-x^2}$ or $100>2000-4x^2$ or $4x^2>1900$ or $x^2>475 \implies x>21$ for integers x. So the only x that works is $x=22$. So the answer is $22+10=32$ Jason is creating a structure out of steel bars. First, he makes a cube. He then connects the midpoints of the faces to form a regular octahedron. He continues by connecting the midpoints of the faces of this octahedron to form another, smaller cube. Find the ratio of the volme of the smaller cube to the volume of the larger cube. *Proposed by: Peter Rowley* **ANSWER:** $\boxed{\frac{1}{27}}$ **SOLUTION:** Suppose Ben builds another triangle $\triangle ABC$ which has sidelengths $AB = 13, BC = 14, CA = 15$. Let D be the point of tangency between the incircle of $\triangle ABC$ and side BC, and let M be the midpoint of BC. The circumcircle of $\triangle ADM$ intersects the circumcircle of $\triangle ABC$ at a point $P \neq A$. If AP intersects BC at Q, find the length of BQ. *Proposed by: Nathan Ramesh* **ANSWER:** $\boxed{42}$ **SOLUTION:** Note that $BD = 6, BM = 7, BC = 14$. From power of a point, we have

$$\begin{aligned} QB \cdot QC &= QA \cdot QP \\ &= QD \cdot QM. \end{aligned}$$

Thus,

$$\begin{aligned} BQ(BQ+14) &= (BQ+6)(BQ+7) \\ 14 \cdot BQ &= 13 \cdot BQ + 42 \\ BQ &= 42. \end{aligned}$$

Music

6. Generally, only frequencies between 20 Hz and 20,000 Hz are considered audible. Nathan has a special LMT clarinet that only plays notes at integer multiples of 2017 Hz. How many different audible notes can Nathan play on his clarinet?

   Proposed by: John Guo

   ANSWER: $\boxed{9}$

   SOLUTION: The largest multiple of 2017 less than 20,000 is $18153 = 2017 \cdot 9$, and the smallest multiple of 2017 greater than 20 is 2017. Between 2017 and 18153, there are 9 multiples of 2017. Thus, the number of different notes Nathan's clarinet can play is $\boxed{9}$.

7. Nathan's chamber group of 6 people have to line up to take a photo. They have heights of $\{62, 65, 65, 67, 69, 70\}$. They must line up left to right with the rule that the heights of 2 people standing next to each other can differ by at most 3. Find the number of ways in which this chamber group can line up from left to right.

   Proposed by: Evan Fang

   ANSWER: $\boxed{16}$

   SOLUTION: For this solution I will refer to people by their heights (so the person with height 62 will be referred to as 62). To distinguish the 65s, I will call one 65a and another 65b.

   If 62 stands on the left, then either 65a or 65b is standing next to him in some order and then we have to have 67 stand next to the last 65 and we can have 69 or 70 next to 67. This gives a total of $2 \cdot 2 = 4$ ways to line up with 62 on the leftmost and 4 on the rightmost due to symmetry.

   If 62 is somewhere in the middle, then 65a and 65b must be standing around him. Furthermore, one of the 65s has to be at the end because the only person left which can stand next to a 65 is 67. So again, lets assume that we are on the left side. Then 67 stands next to the 65 who is not at the end and there are 2 ways again to arrange 69 and 70. This again gives $2 \cdot 2 \cdot 2 = 8$ orderings.

   Thus the answer is $8 + 8 = 16$

8. Mark, who loves both music and math, plays middle A on his clarinet at a frequency A_0. Then, one by one, each one of his m students plays a note one octave above the previous one. Using his math skills, Mark finds that, rounding

to the nearest tenth, $\log_2 A_0 + \log_2 A_1 + ... + \log_2 A_m = 151.8$, where A_n denotes the frequency of the note n octaves above middle A. Given that $\log_2(A_0) = 8.8$, and that $A_n = A_0 \cdot 2^n$ for all positive integers n, how many students does Mark have?

Proposed by: John Guo

ANSWER: $\boxed{10}$

SOLUTION: Note that given the definition of A_m, the series $A_0, A_1, ..., A_m$ is a geometric series with ratio 2. Then, we find that $\log_2 A_0 + \log_2 A_1 + ... + \log_2 A_m$

$= (\log_2 A_0) + (\log_2 A_0 + 1) + (\log_2 A_0 + 2) + ... + (\log_2 A_0 + m)$

$= (m+1)(\log_2 A_0) + 1 + 2 + ... + m = 8.8(m+1) + \dfrac{(m)(m+1)}{2}$.

Seeing as $1, 2, ..., m$ are all whole numbers, the decimal part of 151.8 must come from the term $8.8(m+1)$. Only if the units digit of $(m+1)$ is 1 or 6 will the decimal part of $8.8(m+1) + \dfrac{(m)(m+1)}{2}$ be .8. Thus, a quick guess and check nets us $(m+1) = 11$, so $m = 10$.

9. Janabel numbers the keys on her small piano from 1 to 10. She wants to choose a quintuplet of these keys (a, b, c), such that $a < b < c$ and each of these numbers are pairwise relatively prime. How many ways can she do this?

Proposed by: Evan Fang

ANSWER: $\boxed{21}$

SOLUTION: We use PIE. First, there are $\binom{10}{3} = 120$ ways to choose 3 numbers. Now, we consider every single pair of numbers that are not relatively prime. There are $\binom{5}{2} = 10$ ways to choose numbers such that they share a divisor of 2. There are $\binom{3}{2} = 3$ ways to choose pairs sharing a common divisor of 3 and there are $\binom{2}{2} = 1$ way to choose a pair of numbers sharing a common divisor of 5. In every single one of these pairs, we can choose 8 other numbers to create 8 possible triples per pair. $(10+3+1) \cdot 8 = 112$

Now, we need to add back the triples that we overcounted. First off, this includes all triples with all 3 numbers sharing a common divisor. There are $\binom{5}{3} = 10$ ways to choose 3 sharing a divisor of 2, $\binom{3}{3} = 1$ way to choose numbers containing a divisor of 3. But we also overcounted $(2,3,6)$ once as well as $(2,5,10)$. So we overcounted a total of 13 triples.

Thus, our final answer is $120 - 112 + 13 = 21$

10. Every day, John practices oboe in exact increments of either 0 minutes, 30 minutes, 1 hour, 1.5 hours, or 2 hours. How many possible ways can John practice oboe for a total of 5 hours in the span of 5 consecutive days?

Proposed by: Evan Fang

ANSWER: $\boxed{381}$

SOLUTION: Mulyiply everything by 2; that is John practices in increments of $0, 1, 2, 3, 4$ hours per day and he must practice 10 hours in the next 5 days. This is just the coefficient of x^{10} given by the generating function $(1 + x + x^2 + x^3 + x^4)^5$. This is $(1 - x^5)^5 \cdot \frac{1}{(1-x)^5} = \sum_{k=0}^{5} \binom{k}{5}(-x^5)^k \cdot \sum_{k \geq 0} \binom{k+4}{4} x^k$. With, this we see that the term for x^{10} is $\binom{5}{2}\binom{4}{4} - \binom{5}{1}\binom{9}{4} + \binom{5}{0}\binom{14}{4} = 381$

Games

11. A deck of cards contains 4 suites with 13 numbers in each suit. Evan and Albert are playing a game with a deck of cards. First, Albert draws a card. Evan wins if he draws a card with the same number as Albert's card or the same suit as Albert's card. What is the probability that Evan wins?

    Proposed by: Evan Fang

    ANSWER: $\boxed{\frac{5}{17}}$

    SOLUTION: There are 51 cards left in the deck, 12 cards left in the suit, and 3 cards with the same number. So the probability is $\frac{12+3}{51} = \frac{5}{17}$

12. 2 squares in a square grid are called adjacent if they share a side. In the game of minesweeper, we have a 2017×2017 grid of squares such that each square adjacent to a square which contains a mine is marked (A square with a mine in it is not marked). Also, every square with a mine in it is adjacent to at least one square without a mine in it. Given that there are $5,000$ mines, what is the difference between the greatest and least number of marked squares?

    Proposed by: Evan Fang

    ANSWER: $\boxed{14,999}$

    SOLUTION: Starting in the second column from the left, place a mine in the second square going down. This covers 4 squares, which is the most squares we can cover per mine. Now, in this column, keep placing mines in every two squares until you have reached the 2015th square going down in the column. This gives 672 mines. We can follow this pattern every 2 columns and there are clearly enough columns to make it so that every single mine marks 4 squares. So the maximum is $4 \cdot 5,000$

    Now, for the minimum, first place 2017 mines in the first column. This way, each mine marks exactly one square which is the best we can do. Do this in every other row until you reach the 5,000th mine, which i forced to mark 1 extra square. So the minimum is 5,001.

    So the difference is $3 \cdot 5,000 - 1 = 14,999$

13. There are 2016 stones in a pile. Alfred and Bobby are playing a game where on each turn they can take either a or b stones from the pile where a and b are distinct integers less than or equal to 6. They alternate turns, with Albert going first, and the last person who is able to take a stone wins For example, if $a = 3, b = 6$ and after Alfred's turn there are 2 stones left, then Alfred wins because Bobby is unable to make a move. Let A represent the number of ordered pairs (a, b) for which Alfred has a winning strategy and B represent the number of ordered pairs (a, b) for which Bobby has a winning strategy. Find $A - B$.

    ANSWER: $\boxed{-26}$

    SOLUTION: (Nathan Ramesh) We claim that Bobby wins if the total number of stones is ever a multiple of $a+b$ before Alfred's turn. His winning strategy is simple. If Alfred took a stones on his previous turn, Bobby should take b stones, and if Alfred took b stones, then Bobby should take a stones. Similarly, Alfred wins if the total number of stones is ever a multiple of $a+b$ before Bobby's turn. The only cases in which $a+b$ does not divide 2016 are $(a,b) = (1,4), (2,3), (5,6)$ (and permutations). [list] [*] If $(a,b) = (1,4)$ then Alfred has a winning strategy. Alfred should take 1 stone on his first turn. Then, since $a+b = 5$ divides $2016-1 = 2015$, Alfred can win. [*] If $(a,b) = (2,3)$ then Bobby has a winning strategy. If Alfred took a stones on his previous turn, Bobby should take b stones, and if Alfred took b stones, then Bobby should take a stones. After each of Bobby's turns, the total number of stones leaves a remainder of 1 when divided by 5, so eventually, Alfred will be left with 1 stone and Bobby will win. [*] If $(a,b) = (5,6)$ then Bobby has a winning strategy. If Alfred took a stones on his previous turn, Bobby should take b stones, and if Alfred took b stones, then Bobby should take a stones. After each of Bobby's turns, the total number of stones leaves a remainder of 2 when divided by 11, so eventually, Alfred will be left with 2 stones and Bobby will win. [/list] The only pairs (a,b) in which Alfred has a winning strategy are $(1,4)$ and $(4,1)$ and Bobby has a winning strategy for all other ordered pairs, which there are $6 \cdot 5 - 2 = 28$ of. So the answer is $2 - 28 = -26$

14. For a positive integer n define $f(n)$ to be the number of unordered triples of positive integers (a, b, c) such that

(a) $a, b, c \le n$, and

(b) There exists a triangle ABC with side lengths a, b, c and points D, E, F on line segments AB, BC, CA respectively such that AD, BE, CF all have integer side lengths and $ADEF$ is a parallelogram.

Evan and Albert play a game where they calculate $f(2017)$ and $f(2016)$. Find $f(2017) - f(2016)$.

Proposed by: Evan Fang

ANSWER: $\boxed{1}$

SOLUTION: First, I will prove that the number of unordered triples satisfying the conditions is the same as the number of unordered triples satisfying the following:

$a, b, c < n, \gcd(a, b, c) > 1$ and a, b, c satisfy the triangle inequality.

First, I will prove that this condition is necessary.

By condition 1, we already have $a, b, c < n$. Since there exists a triangle with side lengths a, b, c then we must have that a, b, c satisfy the triangle inequality.

Now, since the parallelogram has integer side lengths this means AD has integer side lengths. WLOG let $AB = c$ and $DB = k$ where k is an integer. Then $AD = c - k$

Now, from $\triangle DBE \ \ \triangle ABC$ we have $\frac{DB}{AB} = \frac{BE}{BC} \implies \frac{k}{c} = \frac{BE}{b} \implies BE = \frac{bk}{c}$

Similarly, we have $DE = \frac{ak}{c} = AF$ which must have an integer side length. Thus $\frac{ak}{c}, \frac{bk}{c}$ must both be integers. Thus, $\frac{c}{\gcd(k,c)}|a$, $\frac{c}{\gcd(k,c)}|b$ and also we clearly have $\frac{c}{\gcd(k,c)}|c$. Since D is on the segment AB by how we defined it we must have $k < c$ and $\frac{c}{\gcd(k,c)} > 1$. Thus, we have proven that our new condition is necessary.

It's also easy to prove that any triple of positive integers satisfying our new condition also satisfies the property. If $gcd(a, b, c) = m$ then let $DB = \frac{c}{m}$ and we can see that $BE = \frac{b}{m}$ and $DE = AF = \frac{a}{m}$ which must all be integers.

Now, our goal is to find the number of unordered triples containing 2017 as their greatest element.

But note that 2017 is prime. Thus if $\gcd(a, b, 2017) > 1$ and $1 < a, b \le 2017$ then clearly $a = b = 2017$. So the only unordered triple containing 2017 is $(2017, 2017, 2017)$ and thus the answer is 1

15. Two players A and B take turns placing counters in squares of an $1 \times n$ board, with A going first. Each turn, players must place a counter in a square does that not share an edge with any square that already has a counter in it. The first player who is unable to make a move loses. Find all $n \le 20$ for which A has a winning strategy.

Proposed by: Nathan Ramesh

ANSWER: $\boxed{(4, 8, 14, 20)}$

SOLUTION: Change the names A and B.

Using standard nimber analysis we have the following: (n, nimber for $1 \times n$): (0, 0) (1, 1) (2, 1) (3, 2) (4, 0) (5, 3) (6, 1) (7, 1) (8, 0) (9, 3) (10, 3) (11, 2) (12, 2) (13, 4) (14, 0) (15, 5) (16, 2) (17, 2) (18, 3) (19, 3) (20, 0) Thus the desired set is $4, 8, 14, 20$