Team Round Solutions

Lexington High School

April 9, 2016

[70] Potpourri

1. Suppose that 20% of a number is 17. Find 20% of 17% of the number.

   Proposed by: Nathan Ramesh

   ANSWER: $\boxed{\dfrac{289}{100}}$

   SOLUTION: Exploit the commutativity of multiplication to see that we want 17% of 17, which is $\frac{289}{100}$.

2. Let A, B, C, D represent the numbers 1 through 4 in some order, with $A \neq 1$. Find the maximum possible value of

   $$\frac{\log_A B}{C+D}.$$

   Here, $\log_A B$ is the unique real number X such that $A^X = B$.

   Proposed by: Peter Rowley

   ANSWER: $\boxed{\dfrac{1}{2}}$

   SOLUTION: We want to maximize B and minimize A, C, and D, so we will have $C = 1$ or $D = 1$. Assuming $C = 1$, we now want to minimize A and D and maximize B so we will have $B = 4$. This leaves the possible values as $(A, D) = (2, 3)$ or $(3, 2)$. The values for these are $\frac{1}{2}$ and $\frac{\log_3 4}{3}$. However, we have $3^{\frac{3}{2}} = \sqrt{27} > 4$, so $\frac{\log_3 4}{3} < \frac{1}{2}$ and the answer is $\frac{1}{2}$.

3. There are six points in a plane, no four of which are collinear. A line is formed connecting every pair of points. Find the smallest possible number of distinct lines formed.

   Proposed by: Peter Rowley

   ANSWER: $\boxed{7}$

   SOLUTION:

4. Let a, b, c be real numbers which satisfy

   $$\frac{2017}{a} = a(b+c)$$
   $$\frac{2017}{b} = b(a+c)$$
   $$\frac{2017}{c} = c(a+b).$$

Find the sum of all possible values of abc.

Proposed by: Evan Fang

ANSWER: $\boxed{\dfrac{2017}{2}}$

SOLUTION: Adding the 3 equations we get
$2ab+2bc+2ca = 2017(\frac{ab+bc+ca}{abc})$ if $ab+bc+ca \neq 0$ then $abc = \frac{2017}{2}$
if $ab+bc+ca = 0$ then $a(b+c) = ab+ac = -bc = \frac{2017}{a}$ so $abc = -2017$
So the sum of the 2 possibilites are $-2017 + \frac{2017}{2} = -\frac{2017}{2}$

5. Let a and b be complex numbers such that $ab+a+b = (a+b+1)(a+b+3)$. Find all possible values of $\frac{a+1}{b+1}$.

   Proposed by: Nathan Ramesh

   ANSWER: $\boxed{-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i}$

   SOLUTION: Let $x = a+1, y = b+1$. Then $xy - 1 = (x+y)^2 - 1 \implies x^2 + xy + y^2 = 0$. Then $\frac{x}{y}$ is a primitive cube root of unity and the rest is trivial computation. The answers are $-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$.

6. Let $\triangle ABC$ be a triangle. Let X, Y, Z be points on lines BC, CA, and AB, respectively, such that X lies on segment BC, B lies on segment AY, and C lies on segment AZ. Suppose that the circumcircle of $\triangle XYZ$ is tangent to lines AB, BC, and CA with center I_A. If $AB = 20$ and $I_AC = AC = 17$ then compute the length of segment BC.

   Proposed by: Nathan Ramesh

   ANSWER: $\boxed{17}$

   SOLUTION: Let I be the incenter and let $F = CI \cap AB$. Then we have $\angle AI_AC = \angle CAI_A = \angle BAI_A$ from which it follows that $AB \parallel CI_A$. Then $\triangle I_ACI \sim \triangle AFI \implies \angle AFI = 90°$. This immediately gives that $\triangle ABC$ is isosceles, hence it follows that $BC = AC = 17$.

7. An ant makes 4034 moves on a coordinate plane, beginning at the point $(0,0)$ and ending at $(2017, 2017)$. Each move consists of moving one unit in a direction parallel to one of the axes. Suppose that the ant stays within the region $|x-y| \leq 2$. Let N be the number of paths the ant can take. Find the remainder when N is divided by 1000.

   Proposed by: Nathan Ramesh

   ANSWER: $\boxed{442}$

   SOLUTION: answer

8. A 10 digit positive integer $\overline{a_9a_8a_7\cdots a_1a_0}$ with a_9 nonzero is called *deceptive* if there exist distinct indices $i > j$ such that $\overline{a_ia_j} = 37$. Find the number of deceptive positive integers.

   Proposed by: Nathan Ramesh

   ANSWER: $\boxed{687}$

   SOLUTION: The number of *deceptive* numbers is equivalent to the following sum, based on casework on where the leftmost 3 is:

$$10^9 - 9^9 + \sum_{i=0}^{7} 8 \cdot 9^i (10^{8-i} - 9^{8-i})$$

. Cancelling powers of 10 and computing mod 8 and mod 125, one can show that this expression is equivalnt to 687 (mod 1000).

SOLUTION: Let $x = a+1, y = b+1$. Then $xy - 1 = (x+y)^2 - 1 \implies x^2 + xy + y^2 = 0$. Then $\frac{x}{y}$ is a primitive cube root of unity and the rest is trivial computation. The answers are $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

9. A circle passing through the points $(2,0)$ and $(1,7)$ is tangent to the y-axis at $(0,r)$. Find all possible values of r.

Proposed by: Nathan Ramesh

ANSWER: $\boxed{4,24}$

SOLUTION: Let $A = (2,0)$ and $B = (1,7)$ for brevity. Extend AB to hit the y-axis at $C = (0,14)$. Let T be the desired point of tangency. We have

$$\begin{aligned} CT^4 &= CA^2 \cdot CB^2 \\ &= (2^2 + 14^2)(1^2 + 7^2) \\ &= 200 \cdot 50 \\ &= 10^4, \end{aligned}$$

so $CT = 10$. The two possible values of r are 14 ± 10, or $4, 24$.

10. An ellipse with major and minor axes 20 and 17, respectively, is inscribed in a square whose diagonals coincide with the axes of the ellipse. Find the area of the square.

Proposed by: Nathan Ramesh

ANSWER: $\boxed{\dfrac{689}{2}}$

SOLUTION: The parametric form of the ellipse is $(10\cos t, 8.5\sin t)$. We have,

$$(\cos^2 t + \sin^2 t)(10^2 + 8.5^2) \geq (10\cos t + 8.5\sin t)^2,$$

which implies that $10\cos t + 8.5\sin t \leq \frac{\sqrt{689}}{2}$. The requested area is $\frac{689}{2}$.

[130] Long Answer

1. [**10**] Find with proof the smallest positive integer k such that every k-element subset of $\{1,2,3...500\}$ contains two distinct elements a,b such that $a+b$ is also an element of the set.

Proposed by: Srinivasan Sathiamurthy

ANSWER: $\boxed{252}$

SOLUTION: The answer is 252. Let S be a subset of $\{1,2,3,\cdots,499,500\}$ such that for any distinct $a,b \in S$, we have $a+b \notin S$. Let m be the maximum element in S. Consider the sets $\{k, m-k\}$ for $k \leq \lceil \frac{m}{2} \rceil - 1$. Each of these sets may have at most one element in S. Additionally, if m is even, it is also possible for $\frac{m}{2}$ to be put in S. Thus, we have

$$|S| \leq 1 + \left(\left\lceil \frac{m}{2} \right\rceil - 1\right) + 1 = \left\lceil \frac{m}{2} \right\rceil + 1 \leq \left\lceil \frac{500}{2} \right\rceil + 1 = 251.$$

Equality holds for $S = \{250, 251, \cdots, 499, 500\}$. The answer is 252.

2. [**15**] Let $\alpha = \frac{\sqrt{5}+1}{2}$. Find all ordered pairs of positive integers (m,n) with $m \neq n$ such that $\{\alpha^m\} = \{\alpha^n\}$. (Here $\{x\}$ denotes the fractional part of x).

Proposed by: Yiming Zheng

ANSWER: $\boxed{n/a}$

SOLUTION:

3. [**30**] The goal of this problem is to show that the maximum area of a polygon with a fixed number of sides and a fixed perimeter is achieved by a regular polygon.

   (a) [**4**] Prove that the polygon with maximum area must be convex. (Hint: If any angle is concave, show that the polygon's area can be increased.)

   (b) [**8**] Prove that if two adjacent sides have different lengths, the area of the polygon can be increased without changing the perimeter.

   (c) [**4**] Prove that the polygon with maximum area is equilateral, that is, has all the same side lengths.

   It is true that when given all four side lengths in order of a quadrilateral, the maximum area is achieved in the unique configuration in which the quadrilateral is cyclic, that is, it can be inscribed in a circle.

   (d) [**8**] Prove that in an equilateral polygon, if any two adjacent angles are different then the area of the polygon can be increased without changing the perimeter.

   (e) [**4**] Prove that the polygon of maximum area must be equiangular, or have all angles equal.

   (f) [**2**] Prove that the polygon of maximum area is a regular polygon.

4. [**35**] Let A be a list of N positive integers sorted least to greatest. Say we are searching the set for an element E. Define *trivial search* as simply searching for the element from the start of A to the end of A. This can be very inefficient for large lists.

   Define *binary search* as a recursive process for sorted lists as such. Compare E to the middle element of A. If E is greater than the middle element, perform this same process on the second half of the sequence. If E is less than the middle element, perform this same process on the first half of the sequence. This continues until the middle element equals E, or the other half is empty.

   For example, let $A = \{1,2,3,4,5,6\}$ and $E = 4$. We first check the range 1 to 6, where the middle element is 3. As $4 > 3$, we only look at the range above this middle element. This range is from 4 to 6, where the middle element is 5. As $4 < 5$, we only look at the range lower than this element. This range is from 4 to 4, where the middle element is 4. As this equals E, we end our search after only three comparisions.

   (a) [**1**] How many comparisions, at worst-case, will be needed with *binary search* for an element E on a sorted list of length 8?

   (b) [**1**] How many comparisions, at worst-case, will be needed with *binary search* for an element E on a sorted list of length 16?

   (c) [**3**] How many comparisions, at worst-case, will be needed with *binary search* for an element E on a sorted list of length N?

   (d) [**3**] Prove that *binary search* will always determine whether or not E is in element in a sorted list A.

   (e) [**5**] Describe a method to determine the number of elements in a sorted list A that are equal to element E.

   Binary search is a powerful tool, and can be used for a number of different problems involving searching for some quantity in an efficient manner. Binary search can even be used on the real numbers to approximate certain values.

   (a) [**7**] Describe a method to approximate $\sqrt{5}$, using binary search on the range [1, 5].

   (b) [**7**] Say a sorted list A contains the first N elements of a geometric series with starting term a and ratio $r > 1$, but with one element removed. Describe a method to use binary search to determine the removed element if you are given a and r.

   (c) [**8**] Say a sorted list A has all of its elements rotated to the left by k elements (a list {1, 2, 3} rotated by 2 becomes {2, 3, 1}). Describe a method to use binary search to determine the value of k.

5. [**40**] Let P be a point and ω be a circle with center O and radius r. We define the **power** of the point P with respect to the circle ω to be $OP^2 - r^2$, and we denote this by $\text{pow}(P,\omega)$. We define the **radical axis** of two circles ω_1 and ω_2 to be the locus of all points P such that $\text{pow}(P,\omega_1) = \text{pow}(P,\omega_2)$. It turns out that the pairwise radical axes of three circles are either concurrent or pairwise parallel. The concurrence point is referred to as the **radical center** of the three circles.

In $\triangle ABC$, let I be the incenter, Γ be the circumcircle, and O be the circumcenter. Let A_1, B_1, C_1 be the point of tangency of the incircle of $\triangle ABC$ with side BC, CA, AB, respectively. Let $X_1, X_2 \in \Gamma$ such that X_1, B_1, C_1, X_2 are collinear in this order. Let M_A be the midpoint of BC, and define ω_A as the circumcircle of $\triangle X_1 X_2 M_A$. Define ω_B, ω_C analogously. The goal of this problem is to show that the radical center of $\omega_A, \omega_B, \omega_C$ lies on line $\overline{OI}$.

(a) [**4**] Let A_1' denote the intersection of B_1C_1 and BC. Show that $\frac{A_1B}{A_1C} = \frac{A_1'B}{A_1'C}$.

(b) [**8**] Prove that A_1 lies on ω_A.

(c) [**6**] Prove that A_1 lies on the radical axis of ω_B and ω_C.

(d) [**13**] Prove that the radical axis of ω_B and ω_C is perpendicular to B_1C_1.

(e) [**2**] Prove that the radical center of $\omega_A, \omega_B, \omega_C$ is the orthocenter of $\triangle A_1B_1C_1$.

(f) [**7**] Conclude that the radical center of $\omega_A, \omega_B, \omega_C$, O, and I are collinear.

Proposed by: Yiming Zheng

ANSWER: $\boxed{n/a}$

SOLUTION: