Guts Round Solutions

Lexington High School

April 8, 2017

8th Annual Lexington Math Tournament - Guts Round - Part 1

Team Name: ______________________________

____________ 1. [**5**] Find all pairs (a,b) of positive integers with $a > b$ and $a^2 - b^2 = 111$.

Proposed by: Peter Rowley

ANSWER: $\boxed{(20,17),(56,55)}$

SOLUTION: $a^2 - b^2 = 111 \Rightarrow (a+b)(a-b) = 111$ $111 = 3 \times 37$ so $(a+b, a-b)$ is either $(37,3)$ or $(111,1)$. This means that $(a,b) = (20,17),(56,55)$.

____________ 2. [**5**] Alice drives at a constant rate of 2017 miles per hour. Find all positive values of x such that she can drive a distance of x^2 miles in a time of x minutes.

Proposed by: Yiming Zheng

ANSWER: $\boxed{\dfrac{2017}{60}}$

SOLUTION:

____________ 3. [**5**] ABC is a right triangle with right angle at B and altitude BH to hypotenuse AC. If $AB = 20$ and $BH = 12$, find the area of triangle $\triangle ABC$.

Proposed by: Peter Rowley

ANSWER: $\boxed{150}$

SOLUTION: As $\triangle ABH$ is a right triangle we know that $AH = 4\sqrt{5}$. Since $\triangle ABH \sim \triangle BCH$ we know that $CH = 8 \times \frac{8}{4\sqrt{5}} = \frac{16}{\sqrt{5}}$, so $BC = \frac{16}{\sqrt{5}} \times \frac{12}{8} = \frac{24}{\sqrt{5}}$. Thus we know that the area of ABC is $\frac{1}{2}(12)\left(\frac{24}{\sqrt{5}}\right) = \frac{144}{\sqrt{5}} = \frac{144\sqrt{5}}{5}$.

8th Annual Lexington Math Tournament - Guts Round - Part 2

Team Name: ______________________________

____________ 4. [**5**] Regular polygons P_1 and P_2 have n_1 and n_2 sides and interior angles x_1 and x_2, respectively. If $\frac{n_1}{n_2} = \frac{7}{5}$ and $\frac{x_1}{x_2} = \frac{15}{14}$, find the ratio of the sum of the interior angles of P_1 to the sum of the interior angles of P_2.

Proposed by: Peter Rowley

ANSWER: $\boxed{\dfrac{3}{2}}$

SOLUTION: The sum of the interior angles are $n_1 x_1$ and $n_2 x_2$ for P_1 and P_2, respectively. Thus, the answer is $\frac{7}{5} \times \frac{15}{14} = \frac{3}{2}$.

____________ 5. [**5**] Joey starts out with a polynomial $f(x) = x^2 + x + 1$. Every turn, he either adds or subtracts 1 from f. What is the probability that after 2017 turns, f has a real root?

Proposed by: Evan Fang

ANSWER: $\boxed{\frac{1}{2}}$

SOLUTION: Say $f(x) = x^2 + x + c$ after 2017 turns. Then f has a real root iff $1 - 4c \geq 0$. Thus we have $c \leq 0$, so we are looking for the probability that Joey subtracts more times than he adds. Clearly, this happens $\frac{1}{2}$ of the time.

____________ 6. [**5**] Find the difference between the greatest and least positive integer values x such that $\sqrt[20]{\lfloor \sqrt[17]{x} \rfloor} = 1$.

Proposed by: Evan Fang

ANSWER: $\boxed{131070}$

SOLUTION: The solution is just to note that $1 \leq \sqrt[17]{x} < 2$ so the answer is $2^{17} - 2 = 131070$.

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8th Annual Lexington Math Tournament - Guts Round - Part 3

Team Name: ______________________________

____________ 7. [**6**] Let $ABCD$ be a square and suppose P and Q are points on sides AB and CD respectively such that $AP/PB = \frac{20}{17}$ and $CQ/QD = \frac{17}{20}$. Suppose that $PQ = 1$. Find the area of square $ABCD$.

Proposed by: Nathan Ramesh

ANSWER: $\boxed{\frac{1369}{1378}}$

SOLUTION: Let the sidelength of the square be $37x$. Then $(3x)^2 + (37x)^2 = 1 \implies 1378x^2 = 1$. We want $(37x)^2 = \frac{1369}{1378}$.

____________ 8. [**6**] If

$$\frac{\sum_{n\geq 0} r^n}{\sum_{n\geq 0} r^{2n}} = \frac{1 + r + r^2 + r^3 + \cdots}{1 + r^2 + r^4 + r^6 + \cdots} = \frac{20}{17},$$

find r.

Proposed by: Evan Fang

ANSWER: $\boxed{\frac{-20 + \sqrt{502}}{2}}$

SOLUTION: We have $\frac{20}{1+r^2} = \frac{17}{1-r} \implies 17r^2 + 20r - 3 = 0$. Applying the quadratic formula we get two solutions $r = \frac{-20\pm\sqrt{502}}{2}$. Since $|r| < 1$ we must have $r = \frac{-20+\sqrt{502}}{2}$.

____________ 9. [**6**] Let $\overline{abc}$ denote the 3 digit number with digits a, b and c. If $\overline{abc}_{10}$ is divisible by 9, what is the probability that $\overline{abc}_{40}$ is divisible by 9?

Proposed by: Evan Fang

ANSWER: $\boxed{\frac{1}{3}}$

SOLUTION: We have $a + b + c \equiv 0 \pmod 9$. Also, $40^2a + 40b + c \equiv 4^2a + 4b + c \equiv 16a + 7b + c \equiv 7a + 7b + c \equiv 6a + 6b \pmod 9$

Thus $9|\overline{abc}_{40} \iff 6a + 6b \equiv 0 \pmod 9 \iff 2(a + b) \equiv 0 \pmod 3 \iff a + b \equiv 0 \pmod 3$

We can bash this out and find that 30 ordered pairs of (a, b) satisfy this condition. Thus the answer is $\frac{30}{90} = \frac{1}{3}$

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8th Annual Lexington Math Tournament - Guts Round - Part 4

Team Name: ______________________________

____________ 10. [**6**] Find the number of factors of 20^{17} that are perfect cubes but not perfect squares.

Proposed by: Evan Fang

ANSWER: $\boxed{54}$

SOLUTION: $20^{17} = 2^{34} \cdot 5^{17}$

In order for a number to be a perfect cube, the powers of each of its prime factors must be a multiple of 3. Thus, 2 can have a power of $0, 3, 6, \ldots, 33$ and 5 can have a power of $0, 3, 6, .., 15$ giving a total of $12 \cdot 6 = 72$ cubes.

Now from this we need to subtract the number of numbers which are perfect squares and perfect cubes, meaning perfect 6th powers. Again, 2 can have a power of $0, 6, \ldots, 30$ and 5 can have a power of $0, 6, 12$ so there are $6 \cdot 3 = 18$ 6th powers.

The answer is $72 - 18 = 54$

____________ 11. [**6**] Find the sum of all positive integers $x \le 100$ such that x^2 leaves the same remainder as x does upon division by 100.

Proposed by: Evan Fang

ANSWER: $\boxed{202}$

SOLUTION: The condition is equivalent to $2^2 \cdot 5^2 = 100 | (x^2 - x) = x(x-1)$. Since $\gcd(x, x-1) = 1$, either $25|x$ or $25|(x-1)$. If $25|x$, it's easy to check that $x = 25, 100$ are the only solutions. If $25|(x-1)$, it's easy to check that $x = 1, 76$ are the only solutions. The sum is $1 + 25 + 76 + 100 = 202$.

____________ 12. [**6**] Find all b for which the base-b representation of 217 contains only ones and zeros.

Proposed by: Nathan Ramesh

ANSWER: $\boxed{2, 6, 216, 217}$

SOLUTION: Clearly either $b|216$ or $b|217$. The latter immediately gives only $b = 217$ as $217 = 7 \cdot 31$. Furthermore, if b is a valid base, then we must have $b^k \le 217 \le 2b^k$ for some k. This gives $217 \ge b^k \ge 109$. If $b|216$, there is still a fair amount of testing that must be done.

- $k = 7$ has one check $b = 2$, which works
- $k = 6$ has no solutions
- $k = 5$ has no solutions
- $k = 4$ has no solutions
- $k = 3$ has one check $b = 6$, which works
- $k = 2$ has one check $b = 12$, which does not work
- $k = 1$ has one check $b = 216$, which works

The final solution set is $2, 6, 216, 217$.

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8th Annual Lexington Math Tournament - Guts Round - Part 5

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____________ 13. [**7**] Two closed disks of radius $\sqrt{2}$ are drawn centered at the points $(1, 0)$ and $(-1, 0)$. Let $\mathscr{P}$ be the region belonging to both disks. Two congruent non-intersecting open disks of radius r have all of their points in $\mathscr{P}$. Find the maximum possible value of r.

Proposed by: Nathan Ramesh

ANSWER: $\boxed{\dfrac{\sqrt{2}}{4}}$

SOLUTION: Clearly we have $1^2 + r^2 = (\sqrt{2} - r)^2$ which gives $r = \frac{1}{2\sqrt{2}}$. (or equivalently, $r = \frac{\sqrt{2}}{4}$.)

____________ 14. [**7**] A rectangle has positive integer side lengths. The sum of the numerical values of its perimeter and area is 2017. Find the perimeter of the rectangle.

Proposed by: Evan Fang

ANSWER: $\boxed{172}$

SOLUTION: Let the lengths be a and b. Then $ab+2a+2b = (a+2)(b+2)-4 = 2017$ so $(a+2)(b+2) = 2021 = 2025-4 = (45+2)(45-2) = 47\cdot 43$. WLOG we let $a+2 = 43 \implies a = 41$ and $b+2 = 47 \implies b = 45$ and so the perimeter is $2\cdot(41+45) = 172$.

____________ 15. [**7**] Find all ordered triples of real numbers (a, b, c) which satisfy

$$\begin{aligned} a+b+c &= 6 \\ a\cdot(b+c) &= 6 \\ (a+b)\cdot c &= 6. \end{aligned}$$

Proposed by: Nathan Ramesh

ANSWER: $\boxed{(3-\sqrt{3},0,3+\sqrt{3}),(3+\sqrt{3},0,3-\sqrt{3}),(3-\sqrt{3},2\sqrt{3},3-\sqrt{3}),(3+\sqrt{3},-2\sqrt{3},3+\sqrt{3})}$

SOLUTION: Let $f(x) = x^2-6x+6 = (x-\alpha)(x-\beta)$. WLOG suppose $a = \alpha$ and $(b+c) = \beta$ and we can permute α and β at the end. There are two cases

- Case 1: $c = \beta, (a+b) = \alpha$. In this case, $ac = \alpha\beta = 6$. Thus it follows that $ab = 0 \implies b = 0$. The solution in this case is $(a, b, c) = (\alpha, 0, \beta)$.
- Case 2: $c = \alpha, (a+b) = \beta$. In this case the solution is $(a, b, c) = (\alpha, \beta-\alpha, \alpha)$.

Solving gives $\alpha, \beta = 3\pm\sqrt{3}$. The solutions are

$$(a,b,c) = (3-\sqrt{3},0,3+\sqrt{3}),(3+\sqrt{3},0,3-\sqrt{3}),(3-\sqrt{3},2\sqrt{3},3-\sqrt{3}),(3+\sqrt{3},-2\sqrt{3},3+\sqrt{3}).$$

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8th Annual Lexington Math Tournament - Guts Round - Part 6

Team Name: ____________________________

____________ 16. [**7**] A four digit positive integer is called *confused* if it is written using the digits 2, 0, 1, and 7 in some order, each exactly one. For example, the numbers 7210 and 2017 are confused. Find the sum of all confused numbers.

Proposed by: Nathan Ramesh

ANSWER: $\boxed{64995}$

SOLUTION: Take averages! The answer is $18(\frac{10}{3}\cdot 1000+\frac{10}{4}\cdot 111) = 60000+45\cdot 111 = 64995$.

____________ 17. [**7**] Suppose $\triangle ABC$ is a right triangle with a right angle at A. Let D be a point on segment BC such that $\angle BAD = \angle CAD$. Suppose that $AB = 20$ and $AC = 17$. Compute AD.

Proposed by: Nathan Ramesh

ANSWER: $\boxed{\dfrac{340\sqrt{2}}{37}}$

SOLUTION: Extend AD to a point E such that BE is parallel to AC. Note that $\triangle BED \sim \triangle CAD$, which gives

$$\begin{aligned} AD &= \frac{AC}{AC+BE}\cdot AE \\ &= \frac{AC}{AC+AB}\cdot AB\sqrt{2} \\ &= \frac{340\sqrt{2}}{37}. \end{aligned}$$

____________ 18. [**7**] Let x be a real number. Find the minimum possible positive value of

$$\frac{|x-20|+|x-17|}{x}.$$

Proposed by: Nathan Ramesh

ANSWER: $\boxed{\frac{3}{20}}$

SOLUTION: Let c be the desired minimum. Define $f(x) = -cx + |x-20| + |x-17|$. By graphical analysis, it is evident that the minimum of f occurs either $x = 17$ or $x = 20$. Note that $f(20) = 3-20c < 3-17c = f(17)$. Thus, if f has at least one zero, then $\min(f) = f(20) = 3-20c \le 0 \implies c \ge \frac{3}{20}$. The minimum possible positive value of c is $\frac{3}{20}$.

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8th Annual Lexington Math Tournament - Guts Round - Part 7

Team Name: ______________________________

____________ 19. [**8**] Find the sum of all real numbers $0 < x < 1$ that satisfy $\{2017x\} = \{x\}$.

Proposed by: Nathan Ramesh

ANSWER: $\boxed{\frac{2015}{2}}$

SOLUTION:

____________ 20. [**8**] Let $a_1, a_2, \cdots, a_{10}$ be real numbers which sum to 20 and satisfy $\{a_i\} < 0.5$ for $1 \le i \le 10$. Find the sum of all possible values of

$$\sum_{1\le i<j\le 10} \lfloor a_i + a_j \rfloor.$$

Here, $\lfloor x \rfloor$ denotes the greatest integer x_0 such that $x_0 \le x$ and $\{x\} = x - \lfloor x \rfloor$.

Proposed by: Evan Fang

ANSWER: $\boxed{810}$

SOLUTION: Note that the condition $\{a_i\} < 0.5$ implies that $\{a_i + a_j\} = \{a_i\} + \{a_j\}$. Thus, we have

$$\begin{aligned}
\sum_{1\le i<j\le 10} \lfloor a_i + a_j \rfloor &= \sum_{1\le i<j\le 10} (a_i + a_j) - \sum_{1\le i<j\le 10} \{a_i + a_j\} \\
&= 180 - \sum_{1\le i<j\le 10} \{a_i + a_j\} \\
&= 180 - \sum_{1\le i<j\le 10} \{a_i\} + \{a_j\} \\
&= 180 - 9\sum_{i=1}^{10} \{a_i\}.
\end{aligned}$$

Let $S = \sum_{i=1}^{10}\{a_i\}$. Since $\sum_{i=1}^{10} a_i$ is an integer, it follows by taking mod 1 that S is an integer as well. Furthermore, since each term of S is strictly less than 0.5, it follows that $S < 10 \cdot 0.5 = 5$. It's easy to check that the possible values of S are $0, 1, 2, 3, 4$. The sum of all possible values of the desired sum is $180 \cdot 5 - 9 \cdot (0+1+2+3+4) = 810$.

____________ 21. [**8**] Compute the remainder when 20^{2017} is divided by 17.

Proposed by: Janabel Xia

ANSWER: $\boxed{3}$

SOLUTION: Note $20^{16} \equiv 3^{16} \equiv (-4)^4 \equiv 1 \pmod{17}$. Then $20^{2017} \equiv 3^{2016} \cdot 3 = 3^{16 \cdot 126} \cdot 3 \equiv 3 \pmod{17}$.

8th Annual Lexington Math Tournament - Guts Round - Part 8

Team Name: ______________________________

______________ 22. [**8**] Let $\triangle ABC$ be a triangle with a right angle at B. Additionally, let M be the midpoint of AC. Suppose the circumcircle of $\triangle BCM$ intersects segment AB at a point $P \neq B$. If $CP = 20$ and $BP = 17$, compute AC.

Proposed by: Nathan Ramesh

ANSWER: $\boxed{2\sqrt{370}}$

SOLUTION: Let O be the circumcenter of $PBCM$. Since $\angle PBC$ is right, it follows that $\angle PMC$ is right as well. Thus, $\triangle APC$ is isosceles. It follows that

$$\begin{aligned}
AC &= \sqrt{AB^2 + BC^2} \\
&= \sqrt{(AP+PB)^2 + CP^2 - BP^2} \\
&= \sqrt{(20+17)^2 + 20^2 - 17^2} \\
&= \sqrt{1480} \\
&= 2\sqrt{370}.
\end{aligned}$$

______________ 23. [**8**] Two vertices on a cube are called *neighbors* if they are distinct endpoints of the same edge. On a cube, how many ways can a nonempty subset S of the vertices be chosen such that for any vertex $v \in S$, at least two of the three neighbors of v are also in S? Reflections and rotations are considered distinct.

Proposed by: Nathan Ramesh

ANSWER: $\boxed{31}$

SOLUTION: Casework

______________ 24. [**8**] Let x be a real number such that $x + \sqrt[4]{5 - x^4} = 2$. Find all possible values of $x\sqrt[4]{5 - x^4}$.

Proposed by: Nathan Ramesh

ANSWER: $\boxed{4 - \sqrt{\frac{21}{2}}}$

SOLUTION: Let $y = \sqrt[4]{5 - x^4}$ for simplicity. Then we have $x + y = 2$ and $x^4 + y^4 = 5$, and we seek to find xy. Let $s = x + y = 2$ and $p = xy$. We have

$$\begin{aligned}
5 &= x^4 + y^4 \\
&= (x^2 + y^2)^2 - 2x^2y^2 \\
&= (s^2 - 2p)^2 - 2p^2 \\
&= (4 - 2p)^2 - 2p^2 \\
&= 2p^2 - 16p + 16.
\end{aligned}$$

We find that $2(p-4)^2 - 21 = 0 \implies p = 4 \pm \sqrt{\frac{21}{2}}$. Notice that if x must be positive because if x were nonpositive we would have $x + y \le y \le \sqrt[4]{5 - x^4} < 2$. Suppose $xy = 4 + \sqrt{\frac{21}{2}}$. Then one of x, y is greater than 2. This contradicts $x + y = 2$. It follows that $p = 4 - \sqrt{\frac{21}{2}}$.

8th Annual Lexington Math Tournament - Guts Round - Part 9

Team Name: ______________________________

____________ 25. [**9**] Let S be the set of the first 2017 positive integers. Find the number of elements $n \in S$ such that

$$\sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor$$

is even.

Proposed by: Evan Fang

ANSWER: $\boxed{1025}$

SOLUTION: Notice that $\lfloor \frac{n}{k} \rfloor$ counts the number of multiples of k less than or equal to n. So each number less than n is counted once for each divisor it has (so 6 would be counted 4 times,for $\lfloor \frac{n}{1} \rfloor, \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{3} \rfloor, \lfloor \frac{n}{6} \rfloor$ if $n \geq 6$)

Now, denote $\tau(n)$ to be the number of divisors of n. Then $\sum_{i=1}^{n} \lfloor \frac{n}{i} \rfloor = \tau(1) + \tau(2) + ... + \tau(n)$. It is well known that only square numbers have an odd number of divisors.

$\tau(1)$ is odd so $\tau(1) + \tau(2), \tau(1) + \tau(2) + \tau(3)$ are also odd because $\tau(2)$ and $\tau(3)$ are even since they are not squares.

Thus, the number of n such that this is even are the n that are in [4,9) U [16,25) U .... U [1936,2017) or

$(3^2 - 2^2) + (5^2 - 4^2) + (7^2 - 6^2) + (9^2 - 8^2) + ... + (45^2 - 44^2) - (45^2 - 2017 + 1) = 5 + 9 + 13 + ... + 89 - 9 = 94 * 11 - 9 = 1025$

____________ 26. [**9**] Let $\{x_n\}_{n \geq 0}$ be a sequence with $x_0 = 0, x_1 = \frac{1}{20}, x_2 = \frac{1}{17}, x_3 = \frac{1}{10}$, and $x_n = \frac{1}{2}(x_{n-2} + x_{n-4})$ for $n \geq 4$. Compute

$$\left\lfloor \frac{1}{x_{2017!} - x_{2017!-1}} \right\rfloor.$$

Proposed by: Nathan Ramesh

ANSWER: $\boxed{170}$

SOLUTION: Denote $N = 2017!$. Let $y_n = x_{2n} - x_{2n-1}$. For convenience set $x_{-1} = 0$. Then $y_n = \frac{1}{2}(y_{n-1} + y_{n-2})$. Defining $z_n = \frac{340}{3} y_n$ is becomes clear that

$$z_n = \sum_{k=0}^{n-1} \left(-\frac{1}{2}\right)^k$$

for $n \geq 1$. Noting that z_n converges to $\frac{2}{3}$ makes it apparent that $z_{N/2} = \frac{2}{3} - \varepsilon$ for some negligibly small positive epsilon. Despite being incredibly small, noticing that epsilon is positive is the key to correctly evaluating the floor. Now, we have

$$y_{N/2} = \frac{1}{170} - \frac{3\varepsilon}{340}$$

from which it follows that

$$\left\lfloor \frac{1}{x_{2017!} - x_{2017!-1}} \right\rfloor = \left\lfloor \frac{1}{\frac{1}{170} - \frac{3\varepsilon}{340}} \right\rfloor = 170.$$

____________ 27. [**9**] Let $ABCDE$ be be a cyclic pentagon. Given that $\angle CEB = 17$, find $\angle CDE + \angle EAB$, in degrees.

Proposed by: Evan Fang

ANSWER: $\boxed{197}$

SOLUTION: Notice $\angle CDE = \angle BDC + \angle BDA + \angle ADE = \frac{CB+BA+AE}{2}$ and $\angle EAB = \angle EAD + \angle DAC + \angle CAB = \frac{ED+DC+CB}{2}$.

Thus $\angle CDE + \angle EAB = \frac{BA+AE+ED+DC+CB+CB}{2} = 180 + \frac{CB}{2} = 180 + \angle CEB = 197$.

8th Annual Lexington Math Tournament - Guts Round - Part 10

Team Name: ______________________________

____________ 28. [**11**] Let $S = \{1, 2, 4, \cdots, 2^{2016}, 2^{2017}\}$. For each $0 \leq i \leq 2017$, let x_i be chosen uniformly at random from the subset of S consisting of the divisors of 2^i. What is the expected number of distinct values in the set $\{x_0, x_1, x_2, \cdots, x_{2016}, x_{2017}\}$?

Proposed by: Nathan Ramesh

ANSWER: $\boxed{\dfrac{2019}{2}}$

SOLUTION: For brevity, denote $P = \{x_0, x_1, \cdots, x_{2017}\}$. For each $0 \le i \le 2017$, let $X_i = 1$ if $2^i \in P$, and $X_i = 0$ otherwise. Note that the expected value $\mathbb{E}[X_i]$ is the probability that $2^i \in P$. The probability that $2^i \notin P$ is equal to

$$\frac{i}{i+1} \cdot \frac{i+1}{i+2} \cdots \frac{2016}{2017} \cdot \frac{2017}{2018} = \frac{i}{2018},$$

since for each $j \ge i$, the probability that $x_j \ne i$ is $\frac{j}{j+1}$. Thus we have $\mathbb{E}[X_i] = 1 - \frac{i}{2017}$. The expected number of distinct values in P is

$$\mathbb{E}[X_0 + X_1 + X_2 + \cdots + X_{2016} + X_{2017}]$$

By linearity of expectation, we have

$$\begin{aligned}
\sum_{i=0}^{2017} \mathbb{E}[X_i] &= \sum_{i=0}^{2017} 1 - \frac{i}{2018} \\
&= 2018 - \frac{1}{2018} \sum_{i=0}^{2017} i \\
&= 2018 - \frac{1}{2018} \cdot \frac{2017 \cdot 2018}{2} \\
&= \frac{2019}{2}.
\end{aligned}$$

____________ 29. [**11**] For positive real numbers a and b, the points $(a,0), (20,17)$ and $(0,b)$ are collinear. Find the minimum possible value of $a+b$.

Proposed by: Nathan Ramesh

ANSWER: $\boxed{37+4\sqrt{85}}$

SOLUTION: For some constant c, we have $20 = c \cdot a$ and $17 = (1-c) \cdot b$. Thus,

$$a + b = \frac{20}{c} + \frac{17}{1-c} \ge \frac{(\sqrt{20}+\sqrt{17})^2}{(c)+(1-c)} = 37 + 4\sqrt{85},$$

where the inequality follows from Titu's Lemma.

____________ 30. [**11**] Find the sum of the distinct prime factors of $2^{36} - 1$.

Proposed by: Anka Hu

ANSWER: $\boxed{266}$

SOLUTION: We have

$$\begin{aligned}
2^{36} - 1 &= (2^{18} - 1)(2^{18} + 1) \\
&= (2^9 - 1)(2^9 + 1)(2^{18} + 2 \cdot 2^9 + 1 - 2^{10}) \\
&= (2^3 - 1)(2^6 + 2^3 + 1)(2^3 + 1)(2^6 - 2^3 + 1)((2^9 + 1)^2 - (2^5)^2) \\
&= 7 \cdot 73 \cdot 9 \cdot 57 \cdot (2^9 - 2^5 + 1)(2^9 + 2^5 + 1) \\
&= 7 \cdot 73 \cdot 3^2 \cdot 3 \cdot 19 \cdot 481 \cdot 545 \\
&= 7 \cdot 73 \cdot 3^3 \cdot 19 \cdot 13 \cdot 37 \cdot 5 \cdot 109,
\end{aligned}$$

so the distinct prime factors are $3, 5, 7, 13, 19, 37, 73, 109$. Adding gives 266.

8th Annual Lexington Math Tournament - Guts Round - Part 11

Team Name: ____________________________

____________ 31. [**13**] There exist two angle bisectors of the lines $y = 20x$ and $y = 17x$ with slopes m_1 and m_2. Find the unordered pair (m_1, m_2).

Proposed by: Nathan Ramesh, Srinivasan Sathiamurthy

ANSWER: $\boxed{\dfrac{339 \pm \sqrt{116290}}{37}}$

SOLUTION: Let $ABCD$ be a rhombus. Then AD is the angle bisector of $\angle BAC$. Call the lines ℓ_1 and ℓ_2, respectively. In the fact above, take

$$\begin{aligned} A &= (0,0) \\ B &= (\sqrt{b^2+1}, a\sqrt{b^2+1}) \\ C &= (\sqrt{a^2+1}, b\sqrt{a^2+1}) \\ D &= (\sqrt{a^2+1} + \sqrt{b^2+1}, a\sqrt{b^2+1} + b\sqrt{a^2+1}) \end{aligned}$$

It follows that the slope of the angle bisector AD is

$$\frac{a\sqrt{b^2+1} + b\sqrt{a^2+1}}{\sqrt{a^2+1} + \sqrt{b^2+1}} = \frac{\sqrt{a^2+1}\sqrt{b^2+1} + ab - 1}{a+b}.$$

Taking $a = 20$ and $b = 17$ gives the slopes

$$\frac{339 \pm \sqrt{116290}}{37}.$$

____________ 32. [**13**] Triangle $\triangle ABC$ has sidelengths $AB = 13, BC = 14, CA = 15$ and orthocenter H. Let Ω_1 be the circle through B and H, tangent to BC, and let Ω_2 be the circle through C and H, tangent to BC. Finally, let $R \neq H$ denote the second intersection of Ω_1 and Ω_2. Find the length AR.

Proposed by: Yiming Zheng

ANSWER: $\boxed{\dfrac{66}{17}}$

SOLUTION: Call the feet of the altitudes D, E, F as shown and let R' be the intersection of the circumcircles of $AEHF$ and ABC. If M is the midpoint of BC, it is well known that the reflection of H over M, say H', is the antipode of A with respect to the circumcircle of ABC. Since $\angle AR'H = \angle AR'H' = 90°$, it follows that R', H, M, H' are collinear. We have

$$MH \cdot MR' = MH' \cdot MR' = MB \cdot MC = MB^2,$$

which implies that BC is tangent to the circumcircle of BHR'. Analogously, one can show that BC is tangent to the circumcircle of CHR'. Thus, $R = R'$. To finish the problem, note that $\triangle ARH \sim \triangle MDH$, so

$$AR = DM \cdot \frac{AH}{HM}.$$

We have $DM = 2$ and Ð $= \frac{3}{4} \cdot 5 = \frac{15}{4}$. Thus $AH = 12 - \frac{15}{4} = \frac{33}{4}$. Finally, recognize that DHM is an 8-15-17 right triangle, so $HM = \frac{17}{4}$. Thus, we have

$$AR = DM \cdot \frac{AH}{HM} = 2 \cdot \frac{33}{17} = \frac{66}{17}.$$

____________ 33. [**13**] For a positive integer n, let $S_n = \{1, 2, 3, \ldots, n\}$ be the set of positive integers less than or equal to n. Additionally, let

$$f(n) = |\{x \in S_n : x^{2017} \equiv x \pmod{n}\}|.$$

Find $f(2016) - f(2015) + f(2014) - f(2013)$.

Proposed by: Evan Fang

ANSWER: $\boxed{139}$

SOLUTION:

Team Name: ______________________________

____________ 34. [**15**] Estimate the value of

$$\sum_{n=1}^{2017} \varphi(n),$$

where $\varphi(n)$ is the number of numbers less than or equal n that are relatively prime to n. If your estimate is E and the correct answer is A, your score for this problem will be

$$\max\left(0, \left\lfloor 15 - 75\frac{|A-E|}{A} \right\rfloor\right).$$

Proposed by: Nathan Ramesh

ANSWER: $\boxed{1237456}$

SOLUTION: See A002088 of OEIS

____________ 35. [**15**] An *up-down* permutation of order n is a permutation σ of $(1,2,3,\cdots n)$ such that $\sigma(i) < \sigma(i+1)$ if and only if i is odd. Denote by P_n the number of up-down permutations of order n. Estimate the value of $P_{20} + P_{17}$. If your estimate is E and the correct answer is A, your score for this problem will be

$$\max\left(0, 16 - \left\lceil \max\left(\frac{A}{E}, \frac{E}{A}\right)\right\rceil\right).$$

Proposed by: Nathan Ramesh

ANSWER: $\boxed{370581053580501}$

SOLUTION: See A000111 of OEIS

____________ 36. [**15**] For positive integers n, *superfactorial* of n, denoted $n\$$, is defined as the product of the first n factorials. In other words, we have

$$n\$ = \prod_{i=1}^{n} (i!).$$

Estimate the number of digits in the product $(20\$) \cdot (17\$)$. If your estimate is E and the correct answer is A, your score for this problem will be

$$\max\left(0, \left\lfloor 15 - \tfrac{1}{2}|A-E| \right\rfloor\right).$$

Proposed by: Nathan Ramesh

ANSWER: $\boxed{260}$

SOLUTION: See A000178 of OEIS