Theme Round

Lexington High School

April 8, 2017

Building

1. Bob wants to build a bridge. This bridge has to be an arch bridge which reaches down on each side 10 feet and crosses a 20 foot gap. If the arch is shown by a semi circle with radius 9, and the bridge is 5 feet wide, how many cubic feet of material does Bob need?

2. Evan constructs a "poly-chain" by connecting regular polygons of side length 1 and having each adjacent polygon share a side. Additionally, Evan only creates a "poly-chain" if the polygons in the chain all have a consecutive number of side lengths. For each "poly-chain", Evan then assigns it an ordered pair (m,n), where m is the number of polygons in the "poly-chain", and n is the number of sides of the largest polygon. Find all ordered pairs (m,n) that correspond to a "poly-chain" with perimeter 17.

3. Ben constructs a triangle ABC such that if M is the midpoint of BC then $AM = BC = 10$. Find the sum of all possible integer valued perimters of $\triangle ABC$.

4. Jason is creating a structure out of steel bars. First, he makes a cube. He then connects the midpoints of the faces to form a regular octahedron. He continues by connecting the midpoints of the faces of this octahedron to form another, smaller cube. Find the ratio of the volme of the smaller cube to the volume of the larger cube.

5. Suppose Ben builds another triangle $\triangle ABC$ which has sidelengths $AB = 13, BC = 14, CA = 15$. Let D be the point of tangency between the incircle of $\triangle ABC$ and side BC, and let M be the midpoint of BC. The circumcircle of $\triangle ADM$ intersects the circumcircle of $\triangle ABC$ at a point $P \neq A$. If AP intersects BC at Q, find the length of BQ.

Music

6. Generally, only frequencies between 20 Hz and 20,000 Hz are considered audible. Nathan has a special LMT clarinet that only plays notes at integer multiples of 2017 Hz. How many different audible notes can Nathan play on his clarinet?

7. Nathan's chamber group of 6 people have to line up to take a photo. They have heights of $\{62, 65, 65, 67, 69, 70\}$. They must line up left to right with the rule that the heights of 2 people standing next to each other can differ by at most 3. Find the number of ways in which this chamber group can line up from left to right.

8. Mark, who loves both music and math, plays middle A on his clarinet at a frequency A_0. Then, one by one, each one of his m students plays a note one octave above the previous one. Using his math skills, Mark finds that, rounding to the nearest tenth, $\log_2 A_0 + \log_2 A_1 + ... + \log_2 A_m = 151.8$, where A_n denotes the frequency of the note n octaves above middle A. Given that $\log_2(A_0) = 8.8$, and that $A_n = A_0 \cdot 2^n$ for all positive integers n, how many students does Mark have?

9. Janabel numbers the keys on her small piano from 1 to 10. She wants to choose a quintuplet of these keys (a, b, c), such that $a < b < c$ and each of these numbers are pairwise relatively prime. How many ways can she do this?

10. Every day, John practices oboe in exact increments of either 0 minutes, 30 minutes, 1 hour, 1.5 hours, or 2 hours. How many possible ways can John practice oboe for a total of 5 hours in the span of 5 consecutive days?

Games

11. A deck of cards contains 4 suites with 13 numbers in each suit. Evan and Albert are playing a game with a deck of cards. First, Albert draws a card. Evan wins if he draws a card with the same number as Albert's card or the same suit as Albert's card. What is the probability that Evan wins?

12. 2 squares in a square grid are called adjacent if they share a side. In the game of minesweeper, we have a 2017×2017 grid of squares such that each square adjacent to a square which contains a mine is marked (A square with a mine in it is not marked). Also, every square with a mine in it is adjacent to at least one square without a mine in it. Given that there are $5,000$ mines, what is the difference between the greatest and least number of marked squares?

13. There are 2016 stones in a pile. Alfred and Bobby are playing a game where on each turn they can take either a or b stones from the pile where a and b are distinct integers less than or equal to 6. They alternate turns, with Albert going first, and the last person who is able to take a stone wins For example, if $a = 3, b = 6$ and after Alfred's turn there are 2 stones left, then Alfred wins because Bobby is unable to make a move. Let A represent the number of ordered pairs (a, b) for which Alfred has a winning strategy and B represent the number of ordered pairs (a, b) for which Bobby has a winning strategy. Find $A - B$.

14. For a positive integer n define $f(n)$ to be the number of unordered triples of positive integers (a, b, c) such that

    (a) $a, b, c \le n$, and

    (b) There exists a triangle ABC with side lengths a, b, c and points D, E, F on line segments AB, BC, CA respectively such that AD, BE, CF all have integer side lengths and $ADEF$ is a parallelogram.

    Evan and Albert play a game where they calculate $f(2017)$ and $f(2016)$. Find $f(2017) - f(2016)$.

15. Two players A and B take turns placing counters in squares of an $1 \times n$ board, with A going first. Each turn, players must place a counter in a square does that not share an edge with any square that already has a counter in it. The first player who is unable to make a move loses. Find all $n \le 20$ for which A has a winning strategy.