Individual Round

Lexington High School

April 8, 2017

1. Find the number of zeroes at the end of 20^{17}.

2. Express $\frac{1}{\sqrt{20}+\sqrt{17}}$ in simplest radical form.

3. John draws a square $ABCD$. On side AB he draws point P so that $\frac{BP}{PA} = \frac{1}{20}$ and on side BC he draws point Q such that $\frac{BQ}{QC} = \frac{1}{17}$. What is the ratio of the area of $\triangle PBQ$ to the area of $ABCD$?

4. Alfred, Bill, Clara, David, and Emily are sitting in a row of five seats at a movie theater. Alfred and Bill don't want to sit next to each other, and David and Emily have to sit next to each other. How many arrangements can they sit in that satisfy these constraints?

5. Alex is playing a game with an unfair coin which has a $\frac{1}{5}$ chance of flipping heads and a $\frac{4}{5}$ chance of flipping tails. He flips the coin three times and wins if he flipped at least one head and one tail. What is the probability that Alex wins?

6. Positive two-digit number $\overline{ab}$ has 8 divisors. Find the number of divisors of the four-digit number $\overline{abab}$.

7. Call a positive integer n *diagonal* if the number of diagonals of a convex n-gon is a multiple of the number of sides. Find the number of diagonal positive integers less than or equal to 2017.

8. There are 4 houses on a street, with 2 on each side, and each house can be colored one of 5 different colors. Find the number of ways that the houses can be painted such that no two houses on the same side of the street are the same color and not all the houses are different colors.

9. Compute
$$|2017-|2016-|2015-|\cdots|3-|2-1||\cdots||||.$$

10. Given points A, B in the coordinate plane, let $A \oplus B$ be the unique point C such that $\overline{AC}$ is parallel to the x-axis and $\overline{BC}$ is parallel to the y-axis. Find the point (x, y) such that
$$((x,y)\oplus(0,1))\oplus(1,0) = (2016,2017)\oplus(x,y).$$

11. In the following subtraction problem, different letters represent different nonzero digits.
$$\begin{array}{r} \text{MATH} \\ -\quad \text{HAM} \\ \hline \text{LMT} \end{array}$$
How many ways can the letters be assigned values to satisfy the subtraction problem?

12. If m and n are integers such that $17n+20m = 2017$, then what is the minimum possible value of $|m-n|$?

13. Let $f(x) = x^4 - 3x^3 + 2x^2 + 7x - 9$. For some complex numbers a, b, c, d, it is true that $f(x) = (x^2 + ax + b)(x^2 + cx + d)$ for all complex numbers x. Find $\frac{a}{b} + \frac{c}{d}$.

14. A positive integer is called an *imposter* if it can be expressed in the form $2^a + 2^b$ where a, b are non-negative integers and $a \neq b$. How many almost positive integers less than 2017 are imposters?

15. Evaluate the infinite sum

$$\sum_{n=1}^{\infty} \frac{n(n+1)}{2^{n+1}} = \frac{1}{2} + \frac{3}{4} + \frac{6}{8} + \frac{10}{16} + \frac{15}{32} + \cdots .$$

16. Each face of a regular tetrahedron is colored either red, green, or blue, each with probability $\frac{1}{3}$. What is the probability that the tetrahedron can be placed with one face down on a table such that each of the three visible faces are either all the same color or all different colors?

17. Let $(k, \sqrt{k})$ be the point on the graph of $y = \sqrt{x}$ that is closest to the point $(2017, 0)$. Find k.

18. Alice is going to place 2016 rooks on a 2016×2016 chessboard where both the rows and columns are labelled 1 to 2016; the rooks are placed so that no two rooks are in the same row or the same column. The value of a square is the sum of its row number and column number. The score of an arrangement of rooks is the sum of the values of all the occupied squares. Find the average score over all valid configurations.

19. Let $f(n)$ be a function defined recursively across the natural numbers such that $f(1) = 1$ and $f(n) = n^{f(n-1)}$. Find the sum of all positive divisors less than or equal to 15 of the number $f(7) - 1$.

20. Find the number of ordered pairs of positive integers (m, n) that satisfy

$$\gcd(m, n) + \operatorname{lcm}(m, n) = 2017.$$

21. Let $\triangle ABC$ be a triangle. Let M be the midpoint of AB and let P be the projection of A onto BC. If $AB = 20$, and $BC = MC = 17$, compute BP.

22. For positive integers n, define the odd parent function, denoted $\operatorname{op}(n)$, to be the greatest positive odd divisor of n. For example, $\operatorname{op}(4) = 1, \operatorname{op}(5) = 5$, and $\operatorname{op}(6) = 3$. Find

$$\sum_{i=1}^{256} \operatorname{op}(i).$$

23. Suppose $\triangle ABC$ has sidelengths $AB = 20$ and $AC = 17$. Let X be a point inside $\triangle ABC$ such that $BX \perp CX$ and $AX \perp BC$. If $|BX^4 - CX^4| = 2017$, the compute the length of side BC.

24. How many ways can some squares be colored black in a 6×6 grid of squares such that each row and each column contain exactly two colored squares? Rotations and reflections of the same coloring are considered distinct.

25. Let $ABCD$ be a convex quadrilateral with $AB = BC = 2$, $AD = 4$, and $\angle ABC = 120°$. Let M be the midpoint of BD. If $\angle AMC = 90°$, find the length of segment CD.