Guts Round

Lexington High School

April 8, 2017

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8th Annual Lexington Math Tournament - Guts Round - Part 1

Team Name: ______________________________

____________ 1. [**5**] Find all pairs (a,b) of positive integers with $a > b$ and $a^2 - b^2 = 111$.

____________ 2. [**5**] Alice drives at a constant rate of 2017 miles per hour. Find all positive values of x such that she can drive a distance of x^2 miles in a time of x minutes.

____________ 3. [**5**] ABC is a right triangle with right angle at B and altitude BH to hypotenuse AC. If $AB = 20$ and $BH = 12$, find the area of triangle $\triangle ABC$.

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____________ 4. [**5**] Regular polygons P_1 and P_2 have n_1 and n_2 sides and interior angles x_1 and x_2, respectively. If $\frac{n_1}{n_2} = \frac{7}{5}$ and $\frac{x_1}{x_2} = \frac{15}{14}$, find the ratio of the sum of the interior angles of P_1 to the sum of the interior angles of P_2.

____________ 5. [**5**] Joey starts out with a polynomial $f(x) = x^2 + x + 1$. Every turn, he either adds or subtracts 1 from f. What is the probability that after 2017 turns, f has a real root?

____________ 6. [**5**] Find the difference between the greatest and least positive integer values x such that $\sqrt[20]{\lfloor \sqrt[17]{x} \rfloor} = 1$.

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________ 7. [**6**] Let $ABCD$ be a square and suppose P and Q are points on sides AB and CD respectively such that $AP/PB = \frac{20}{17}$ and $CQ/QD = \frac{17}{20}$. Suppose that $PQ = 1$. Find the area of square $ABCD$.

________ 8. [**6**] If

$$\frac{\sum_{n\geq 0} r^n}{\sum_{n\geq 0} r^{2n}} = \frac{1+r+r^2+r^3+\cdots}{1+r^2+r^4+r^6+\cdots} = \frac{20}{17},$$

find r.

________ 9. [**6**] Let $\overline{abc}$ denote the 3 digit number with digits a, b and c. If $\overline{abc}_{10}$ is divisible by 9, what is the probability that $\overline{abc}_{40}$ is divisible by 9?

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________ 10. [**6**] Find the number of factors of 20^{17} that are perfect cubes but not perfect squares.

________ 11. [**6**] Find the sum of all positive integers $x \leq 100$ such that x^2 leaves the same remainder as x does upon division by 100.

________ 12. [**6**] Find all b for which the base-b representation of 217 contains only ones and zeros.

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________ 13. [**7**] Two closed disks of radius $\sqrt{2}$ are drawn centered at the points $(1,0)$ and $(-1,0)$. Let $\mathscr{P}$ be the region belonging to both disks. Two congruent non-intersecting open disks of radius r have all of their points in $\mathscr{P}$. Find the maximum possible value of r.

________ 14. [**7**] A rectangle has positive integer side lengths. The sum of the numerical values of its perimeter and area is 2017. Find the perimeter of the rectangle.

________ 15. [**7**] Find all ordered triples of real numbers (a, b, c) which satisfy

$$a+b+c=6$$
$$a\cdot(b+c)=6$$
$$(a+b)\cdot c=6.$$

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_____________ 16. [**7**] A four digit positive integer is called *confused* if it is written using the digits $2, 0, 1$, and 7 in some order, each exactly one. For example, the numbers 7210 and 2017 are confused. Find the sum of all confused numbers.

_____________ 17. [**7**] Suppose $\triangle ABC$ is a right triangle with a right angle at A. Let D be a point on segment BC such that $\angle BAD = \angle CAD$. Suppose that $AB = 20$ and $AC = 17$. Compute AD.

_____________ 18. [**7**] Let x be a real number. Find the minimum possible positive value of

$$\frac{|x-20|+|x-17|}{x}.$$

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_____________ 19. [**8**] Find the sum of all real numbers $0 < x < 1$ that satisfy $\{2017x\} = \{x\}$.

_____________ 20. [**8**] Let $a_1, a_2, \cdots, a_{10}$ be real numbers which sum to 20 and satisfy $\{a_i\} < 0.5$ for $1 \le i \le 10$. Find the sum of all possible values of

$$\sum_{1\le i<j\le 10} \lfloor a_i + a_j \rfloor.$$

Here, $\lfloor x \rfloor$ denotes the greatest integer x_0 such that $x_0 \le x$ and $\{x\} = x - \lfloor x \rfloor$.

_____________ 21. [**8**] Compute the remainder when 20^{2017} is divided by 17.

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_____________ 22. [**8**] Let $\triangle ABC$ be a triangle with a right angle at B. Additionally, let M be the midpoint of AC. Suppose the circumcircle of $\triangle BCM$ intersects segment AB at a point $P \ne B$. If $CP = 20$ and $BP = 17$, compute AC.

_____________ 23. [**8**] Two vertices on a cube are called *neighbors* if they are distinct endpoints of the same edge. On a cube, how many ways can a nonempty subset S of the vertices be chosen such that for any vertex $v \in S$, at least two of the three neighbors of v are also in S? Reflections and rotations are considered distinct.

_____________ 24. [**8**] Let x be a real number such that $x + \sqrt[4]{5-x^4} = 2$. Find all possible values of $x\sqrt[4]{5-x^4}$.

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______ 25. [**9**] Let S be the set of the first 2017 positive integers. Find the number of elements $n \in S$ such that

$$\sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor$$

is even.

______ 26. [**9**] Let $\{x_n\}_{n\geq 0}$ be a sequence with $x_0 = 0, x_1 = \frac{1}{20}, x_2 = \frac{1}{17}, x_3 = \frac{1}{10}$, and $x_n = \frac{1}{2}(x_{n-2} + x_{n-4})$ for $n \geq 4$. Compute

$$\left\lfloor \frac{1}{x_{2017!} - x_{2017!-1}} \right\rfloor.$$

______ 27. [**9**] Let $ABCDE$ be be a cyclic pentagon. Given that $\angle CEB = 17$, find $\angle CDE + \angle EAB$, in degrees.

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______ 28. [**11**] Let $S = \{1, 2, 4, \cdots, 2^{2016}, 2^{2017}\}$. For each $0 \leq i \leq 2017$, let x_i be chosen uniformly at random from the subset of S consisting of the divisors of 2^i. What is the expected number of distinct values in the set $\{x_0, x_1, x_2, \cdots, x_{2016}, x_{2017}\}$?

______ 29. [**11**] For positive real numbers a and b, the points $(a, 0), (20, 17)$ and $(0, b)$ are collinear. Find the minimum possible value of $a + b$.

______ 30. [**11**] Find the sum of the distinct prime factors of $2^{36} - 1$.

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____________ 31. [**13**] There exist two angle bisectors of the lines $y = 20x$ and $y = 17x$ with slopes m_1 and m_2. Find the unordered pair (m_1, m_2).

____________ 32. [**13**] Triangle $\triangle ABC$ has sidelengths $AB = 13, BC = 14, CA = 15$ and orthocenter H. Let Ω_1 be the circle through B and H, tangent to BC, and let Ω_2 be the circle through C and H, tangent to BC. Finally, let $R \neq H$ denote the second intersection of Ω_1 and Ω_2. Find the length AR.

____________ 33. [**13**] For a positive integer n, let $S_n = \{1, 2, 3, \ldots, n\}$ be the set of positive integers less than or equal to n. Additionally, let

$$f(n) = |\{x \in S_n : x^{2017} \equiv x \pmod{n}\}|.$$

Find $f(2016) - f(2015) + f(2014) - f(2013)$.

Warning: The next round is the final round and will consist of three estimation problems.

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____________ 34. [**15**] Estimate the value of

$$\sum_{n=1}^{2017} \varphi(n),$$

where $\varphi(n)$ is the number of numbers less than or equal n that are relatively prime to n. If your estimate is E and the correct answer is A, your score for this problem will be

$$\max\left(0, \left\lfloor 15 - 75\frac{|A-E|}{A}\right\rfloor\right).$$

____________ 35. [**15**] An *up-down* permutation of order n is a permutation σ of $(1, 2, 3, \cdots n)$ such that $\sigma(i) < \sigma(i+1)$ if and only if i is odd. Denote by P_n the number of up-down permutations of order n. Estimate the value of $P_{20} + P_{17}$. If your estimate is E and the correct answer is A, your score for this problem will be

$$\max\left(0, 16 - \left\lceil \max\left(\frac{A}{E}, \frac{E}{A}\right)\right\rceil\right).$$

____________ 36. [**15**] For positive integers n, *superfactorial* of n, denoted $n\$$, is defined as the product of the first n factorials. In other words, we have

$$n\$ = \prod_{i=1}^{n} (i!).$$

Estimate the number of digits in the product $(20\$) \cdot (17\$)$. If your estimate is E and the correct answer is A, your score for this problem will be

$$\max\left(0, \left\lfloor 15 - \tfrac{1}{2}|A-E|\right\rfloor\right).$$