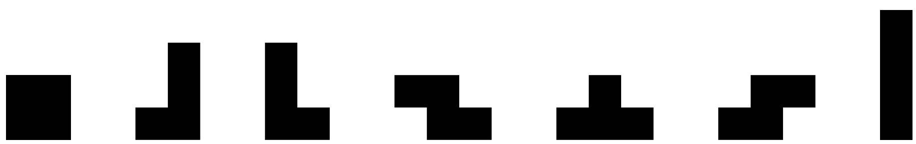
Theme Round Solutions

LMT "Fall"

December 17, 2022

Tetris

Tetris is a Soviet block game developed in 1984, probably to torture misbehaving middle school children. Nowadays, Tetris is a game that people play for fun, and we even have a mini-event featuring it, but it shall be used on this test for its original purpose. The 7 Tetris pieces, which will be used in various problems in this theme, are as follows:



1. **[6]** Each piece has area 4. Find the sum of the perimeters of each of the 7 Tetris pieces.

   Proposed by Jeff Lin

   Solution. $\boxed{68}$

   We count up all the perimeters and get an answer of $\boxed{68}$ □

2. **[8]** In a game of Tetris, Qinghan places 4 pieces every second during the first 2 minutes, and 2 pieces every second for the remainder of the game. By the end of the game, her average speed is 3.6 pieces per second. Find the duration of the game in seconds.

   Proposed by Hannah Shen

   Solution. $\boxed{150}$

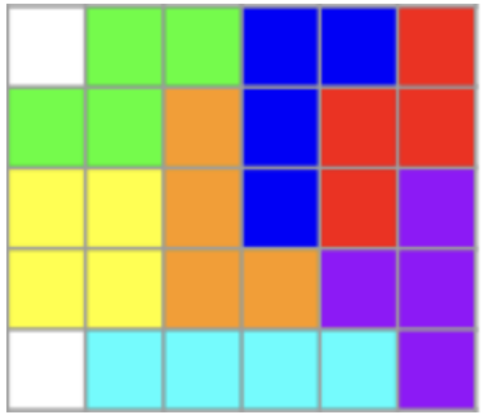
   Call t the duration of the game in seconds. During this game, she will have placed $120*4+(t-120)*2 = 2t+240$ pieces. Thus, $3.6 = \frac{2t+240}{t}$; solving, $t = \boxed{150}$. □

3. **[10]** Jeff takes all 7 different Tetris pieces and puts them next to each other to make a shape. Each piece has an area of 4. Find the least possible perimeter of such a shape.

   Proposed by Jeff Lin and Hannah Shen

   Solution. $\boxed{22}$

   Since each tetris piece has an area of 4, the total area of all 7 pieces will be 28. The largest area that could be enclosed with a perimeter of 20 is a 5 by 5 square (since tetris pieces are grid-like), with an area of 25. Since our area is greater than that, the minimal perimeter must be over 20. Furthermore, the perimeter must be an even number, since any combination of unit squares will have one side facing up for every side facing down, and one side facing left for every side facing right, and Tetris pieces are made of unit squares, so, a perimeter of 21 is not possible. However, we can achieve a perimeter of 22, so 22 is the minimal perimeter.



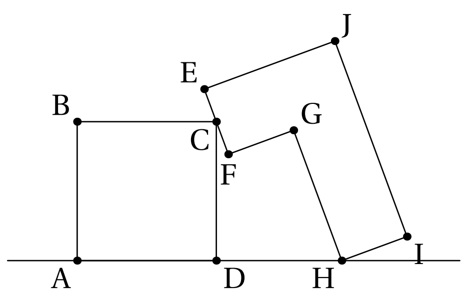
□

4. **[12]** Qepsi is playing Tetris, but little does she know: the latest update has added realistic physics! She places two blocks, which form the shape below. Tetrominoes $ABCD$ and $EFGHIJ$ are both formed from 4 squares of side length 1. Given that $CE = CF$, the distance from point I to the line AD can be expressed as $\frac{A\sqrt{B}-C}{D}$. Find $1000000A+10000B+100C+D$.

   Proposed by Hannah Shen

   Solution. $\boxed{5130829}$

   Label $\angle GHD = \theta$. By drawing some perpendiculars and forming many similar right triangles, we find that $2 = 2\sin\theta - \cos\theta + \frac{1}{2}\sin\theta$. Turning this into a quadratic and solving, we find that $\cos\theta$, which is the height of I, can be expressed as $\frac{5\sqrt{13}-8}{29}$, $\implies \boxed{5130829}$ □



5. **[14]** Using the following tetrominoes:

   Proposed by Hannah Shen

   Solution. $\boxed{32}$

   Call these three sorts of tetrominos S, T, and Z, respectively. This shape fits exactly 6 tetrominos. Note that there must be an even number of T's, with one pointing up and the other pointing down – otherwise, the top and bottom row cannot have equal length. We can do casework on the number of T's.

   If there are 0 T's, there is only 1 way to tile, using all S's.

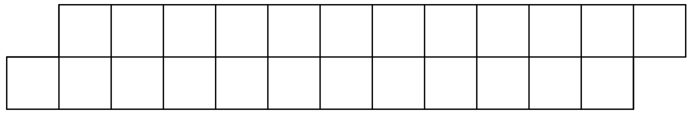
   If there are 2 T's, there are $\binom{6}{2} = 15$ ways to choose their positions in the sequence. Note that the first T must point up and the second down, and that after the 2 T's are placed, there is only 1 way to place the remaining S's and Z's.

   If there are 4 T's, there are $\binom{6}{4} = 15$ ways to choose their positions in the sequence. Similarly, the first and third T must point up, the second and fourth T must point down, and all there is only 1 way to place the remaining S's and Z's.

   If there are 6 T's, there is only 1 way to tile.

   Hence, our total is $1+15+15+1 = \boxed{32}$. □

Find the number of ways to tile the shape below, with rotation allowed, but reflection disallowed:



World Cup

The World Cup, featuring 17 teams from Europe and South America, as well as 15 other teams that honestly don’t have a chance, is a soccer tournament that is held once every four years. As we speak, Croatia and Morocco are locked in a battle that has no significance whatsoever on the winner, but if you would like live score updates nonetheless, feel free to ask your proctor, who has no obligation whatsoever to provide them.

1. **[6]** During the group stage of the World Cup, groups of 4 teams are formed. Every pair of teams in a group play each other once. Each team earns 3 points for each win and 1 point for each tie. Find the greatest possible sum of the points of each team in a group.

   Proposed by Muztaba Syed

   Solution. $\boxed{18}$

   There are $\binom{4}{2} = 6$ games played, and each contributes a maximum of 3 points, for a total of $\boxed{18}$. □

2. **[8]** In the semi-finals of the World Cup, the ref is bad and lets $11^2 = 121$ players per team go on the field at once. For a given team, one player is a goalie, and every other player is either a defender, midfielder, or forward. There is at least one player in each position. The product of the number of defenders, midfielders, and forwards is a mulitple of 121. Find the number of ordered triples (number of defenders, number of midfielders, number of forwards) that satisfy these conditions.

   Proposed by Muztaba Syed

   Solution. $\boxed{135}$

   Let the number of defenders be d, midfielders m, and forwards f. None of d, m, f are greater than 121, so 2 of them are divisible by 11. This implies that the third is $10 \mod 11$, WLOG let it be f (we multiply by 3 at the end). Now let $d = 11a, m = 11b, f = 11c - 1$. Then $11a + 11b + 11c = 121$. We have that $a + b + c = 11$ and they are all nonzero. This is equivalent to $a' + b' + c' = 8$ where they can be 0. This is the number of ways to arrange 8 stars and 2 bars for a total of $\binom{10}{2} = 45$. Multiplying by the 3 from earlier gives $\boxed{135}$. □

3. **[10]** Messi is playing in a game during the Round of 16. On rectangular soccer field $ABCD$ with $AB = 11$, $BC = 8$, points E and F are on segment BC such that $BE = 3$, $EF = 2$, and $FC = 3$. If the distance between Messi and segment EF is less than 6, he can score a goal. The area of the region on the field where Messi can score a goal is $a\pi + \sqrt{b} + c$, where a, b, and c are integers. Find $10000a + 100b + c$.

   Proposed by Muztaba Syed

   Solution. $\boxed{84312}$

   The area consists of 2 right triangles with legs 3 and $3\sqrt{3}$ in the corners which contributes $9\sqrt{3} = \sqrt{243}$. Then there are 2 sectors with an angle of 30° and radius 6, which contributes $2 \cdot 6^2\pi \cdot \frac{30}{360} = 6\pi$. Finally there is a rectangle with side lengths 2 and 6 which contributes 12, giving a final answer of $6\pi + \sqrt{243} + 12 \implies \boxed{84312}$. □

4. **[12]** The workers are building the World Cup stadium for the 2022 World Cup in Qatar. It would take 1 worker working alone 4212 days to build the stadium. Before construction started, there were 256 workers. However, each day after construction, 7 workers disappear. Find the number of days it will take to finish building the stadium.

   Proposed by Muztaba Syed

Solution. $\boxed{24}$

Let d be the number of days. We want

$$\frac{\frac{1}{2}(519-7d)d}{4212}=1.$$

We see that $d = \boxed{24}$ is a solution. □

5. **[14]** In the penalty kick shootout, 2 teams each get 5 attempts to score. The teams alternate shots and the team that scores a greater number of times wins. At any point, if it’s impossible for one team to win, even before both teams have taken all 5 shots, the shootout ends and no more shots are taken. If each team does take all 5 shots and afterwards the score is tied, the shootout enters sudden death, where teams alternate taking shots until one team has a higher score while both teams have taken the same number of shots.

If each shot has a $\frac{1}{2}$ chance of scoring, the expected number of times that any team scores can be written as $\frac{A}{B}$ for relatively prime positive integers A and B. Find $1000A+B$.

Proposed by Derek Zhao

Solution. $\boxed{321064}$

Using linearity of expectations, the expected number of goals made is the sum of the probabilities of each shot being made, which is half of the probability that the shot is attempted. The first three shots for each team must be taken.

The 4th shot from the first team can only not appear if the score is 3-0 or 0-3, so the 4th shot has probability $\frac{31}{32}$ of appearing.

The 4th shot from the second team does not appear if the first team has 3 more points than the second or the second has 2 more points than the first. This includes "ghost shots" which break the regulation rules. Therefore, this shot has $\frac{7}{8}$ of appearing.

The 5th shot from the first team does not appear if either team has 2 more points than the other including ghost shots. The probability that this shot appears is $\frac{91}{128}$.

The 5th shot from the second team does not appear if the first team has 2 more points than the seconds or the second has 1 more point than the first including "ghost shots." $\frac{63}{128}$.

There is a $\frac{63}{256}$ chance of sudden death. Assuming sudden death happens, it is expected that 2 goals are made from geometric series formula. $\frac{63}{128}$.

Therefore the expected number is

$$\frac{1}{2}\cdot(6+\frac{31}{32}+\frac{7}{8}+\frac{91}{128}+\frac{63}{128})+\frac{63}{128}=\frac{321}{64}$$

□

Ephram

Ephram Chun is a senior and math captain at Lexington High School. He is well-loved by the freshmen, who seem to only listen to him. Other than being the father figure that the freshmen never had, Ephram is also part of the Science Bowl and Science Olympiad teams along with being part of the highest orchestra LHS has to offer. His many hobbies include playing soccer, volleyball, and the many forms of chess. We hope that he likes the questions that we've dedicated to him!

1. **[6]** Ephram is scared of freshmen boys. How many ways can Ephram and 4 distinguishable freshmen boys sit together in a row of 5 chairs if Ephram does not want to sit between 2 freshmen boys?

   Proposed by Brandon Ni

   Solution. $\boxed{48}$

   Note the given condition implies that Ephram must sit at the ends of the row. There are thus 2 places for Ephram to sit, and the remaining 4 freshman can be placed in $4! = 24$ ways. Our final answer is $2 \cdot 24 = \boxed{48}$. ☐

2. **[8]** Ephram, who is a chess enthusiast, is trading chess pieces on the black market. Pawns are worth \$100, knights are worth \$515, and bishops are worth \$396. Thirty-four minutes ago, Ephram made a fair trade: 5 knights, 3 bishops, and 9 rooks for 8 pawns, 2 rooks, and 11 bishops. Find the value of a rook, in dollars.

   Proposed by Michael Han

   Solution. $\boxed{199}$

   The question gives us the equation

   $$5 \cdot 515 + 3 \cdot 396 + 9x = 8 \cdot 100 + 2x + 11 \cdot 396.$$

   We thus have that

   $$\begin{aligned} 2575 + 7x &= 800 + 8 \cdot (400 - 4) \\ 2575 + 7x &= 800 + 3200 - 32 \\ 2607 + 7x &= 4000 \\ 7x &= 1393 \\ x &= \boxed{199}. \end{aligned}$$

   ☐

3. **[10]** Ephram is kicking a volleyball. The height of Ephram's kick, in feet, is determined by

   $$h(t) = -\frac{p}{12}t^2 + \frac{p}{3}t,$$

   where p is his kicking power and t is the time in seconds. In order to reach the height of 8 feet between 1 and 2 seconds, Ephram's kicking power must be between reals a and b. Find is $100a + b$.

   Proposed by Brandon Ni

   Solution. $\boxed{2432}$

   The vertex of the parabola is at $t = 2$, which means that $h(2)$ must be at least 8. This implies that $p \geq 24$. Then time between 1 and 2 seconds condition implies that $h(1) \leq 24$, or $p \leq 32$. Thus, the answer is $100 * 24 + 32 = \boxed{2432}$. ☐

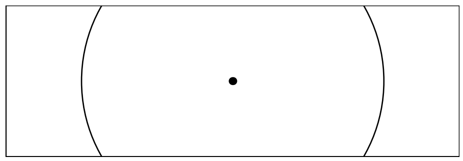
4. **[12]** Disclaimer: No freshmen were harmed in the writing of this problem.

   Ephram has superhuman hearing: He can hear sounds up to 8 miles away. Ephram stands in the middle of a 8 mile by 24 mile rectangular grass field. A freshman falls from the sky above a point chosen uniformly and randomly on the grass field. The probability Ephram hears the freshman bounce off the ground is P%. Find P rounded to the nearest integer.

   Proposed by Brandon Ni

Solution. $\boxed{64}$

The area that Ephram can reach is a circle of radius 8 that shares the same center as the goal. But, the top and bottom section of the goal is cutoff by the crossbar and the ground. Notice that the area can be split into 2 triangles and 2 sixty-degree sectors. The area of Ephram's reach is $2\cdot\frac{1}{2}\cdot 8\cdot 4\sqrt{3}+\frac{1}{3}\cdot 8^2\cdot\pi = 32\sqrt{3}+\frac{64}{3}\pi$. So, the probability is equal to $\frac{32\sqrt{3}+\frac{64}{3}\pi}{8\cdot 24}=\frac{\sqrt{3}}{6}+\frac{\pi}{9}$, which is rounds to $\boxed{64}$%. □



5. **[14]** Ephram and Brandon are playing a version of chess, sitting on opposite sides of a 6×6 board. Ephram has 6 white pawns on the row closest to himself, and Brandon has 6 black pawns on the row closest to himself. During each player's turn, their only legal move is to move one pawn one square forward towards the opposing player. Pawns cannot move onto a space occupied by another pawn. Players alternate turns, and Ephram goes first (of course). Players take turns until there are no more legal moves for the active player, at which point the game ends. Find the number of possible positions the game can end in.

Proposed by Brandon Ni

Solution. $\boxed{1751}$

Note that once one of Ephram's pawns is done moving Brandon's corresponding pawn must go in front of it and therefore has a fixed position. Thus, we shall only consider Ephram's pawns. Each pawn can move between 1 and 4 times: this motivates us to use generating functions. Since the pawns in each column must move 4 times and there are 6 columns, each side takes $\frac{4\cdot 6}{2}=12$ moves. We thus wan to find the coefficient of x^12 in

$$(1+x+x^2+x^3+x^4)^6.$$

We can express this as

$$\left(\frac{1-x^5}{1-x}\right)^6,$$

by using the difference of powers formula. Note that the numerator is equivalent to

$$1+6x^5+15x^{10}+20x^{15}+15x^{20}+6x^{25}+x^{30}$$

by Binomial Theorem. The denominator can be expressed as

$$\frac{1}{(1-x)^6}=(1-x)^{-6}$$

so by Binomial theorem the denominator is equal to

$$1+\frac{-6}{1!}(-x)+\frac{-6\cdot-7}{2!}(-x)^2+\frac{-6\cdot-7\cdot-8}{3!}(-x)^3+\dots=1+\binom{6}{1}x+\binom{7}{2}x^2+\binom{8}{3}x+\dots$$

Thus we wish to find the coefficient of x^12 in

$$\left(1+6x^5+15x^{10}+20x^{15}+15x^{20}+6x^{25}+x^{30}\right)\left(1+\binom{6}{1}x+\binom{7}{2}x^2+\binom{8}{3}x+\dots\right),$$

which is

$$15\binom{7}{2}-6\binom{12}{7}+\binom{17}{12}=\boxed{1751}.$$

□