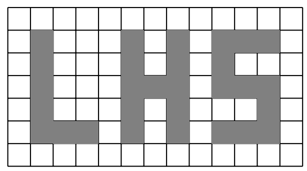
Speed Round Solutions

LMT "Fall"

December 17, 2022

1. **[6]** Each box represents 1 square unit. Find the area of the shaded region.



Proposed by Hannah Shen

Solution. $\boxed{18}$

We count 16 unit squares and two parallelograms, each of which also have area 1, so the total shaded area is $16+2*1=\boxed{18}$. □

2. **[6]** Evaluate $(3^3)\sqrt{5^2-2^4}-5\cdot 9$.

Proposed by Brandon Ni

Solution. $\boxed{36}$

Compute the expression gives us $27*\sqrt{9}-45=81-45=\boxed{36}$ □

3. **[6]** Find the last two digits of 21^3.

Proposed by Jeff Lin

Solution. $\boxed{61}$

$21^3=9261$, so the answer is $\boxed{61}$ □

4. **[6]** Let L, M, and T be distinct prime numbers. Find the least possible odd value of $L+M+T$.

Proposed by Aidan Duncan

Solution. $\boxed{15}$

None of the primes can be 2 because 2+ odd prime + odd prime is even. Therefore, we need the next 3 smallest odd primes, which are $3,5$, and 7 which sum to 15. □

5. **[6]** Two circles have areas that sum to 20π and diameters that sum to 12. Find the radius of the smaller circle.

Proposed by Jeff Lin

Solution. $\boxed{2}$

$x+y=6$, $x^2+y^2=20$. Solve. $x=\boxed{2}$, $y=4$. □

6. **[6]** Zach and Evin each independently choose a date in the year 2022, uniformly and randomly. The probability that at least one of the chosen dates is December 17, 2022 can be expressed as $\frac{A}{B}$ for relatively prime positive integers A and B. Find A.

   Proposed by Evin Liang

   Solution. $\boxed{729}$

   Use PIE. There are 365 days in 2022. $\frac{1}{365}+\frac{1}{365}-\frac{1}{365^2}=\frac{2\cdot 365-1}{365^2}=\frac{729}{365^2}$. Because $729=3^6$, and $\gcd(3^6, 365^2)=1$, we get that $a=\boxed{729}$. □

7. **[6]** Let L be a list of 2023 real numbers with median m. When any two numbers are removed from L, its median is still m. Find the greatest possible number of distinct values in L.

   Proposed by Muztaba Syed

   Solution. $\boxed{2021}$

   After removing 2 values the median can be any of the 3 middle values. This means these are all the same, so there is a maximum of $2023-2=\boxed{2021}$ distinct values. □

8. **[6]** Some children and adults are eating a delicious pile of sand. Children comprise 20% of the group and combined, they consume 80% of the sand. Given that on average, each child consumes N pounds of sand and on average, each adult consumes M pounds of sand, find $\frac{N}{M}$.

   Proposed by Hannah Shen

   Solution. $\boxed{16}$

   Assume there are 20 children and 80 adults, splitting 100 pounds of sand. Then, each child will eat $\frac{80}{20}=4$ pounds, whereas each adult will eat $\frac{20}{80}=\frac{1}{4}$ pounds. Hence, $\frac{N}{M}=\frac{4}{1/4}=\boxed{16}$. □

9. **[6]** An integer N is chosen uniformly and randomly from the set of positive integers less than 100. The expected number of digits in the base-10-representation of N can be expressed as $\frac{A}{B}$ for relatively prime positive integers A and B. Find $1000A+B$.

   Proposed by Hannah Shen

   Solution. $\boxed{21011}$

   There are 9 values of N for which N would have 1 digit: $1,2,3,\dots 9$. There are 90 values of N for which N would have 2 digits: $10,11,12,\dots 99$. Hence, the expected value is $\frac{9*1+90*2}{99}=\frac{21}{11}$, so $1000A+B=\boxed{21011}$. □

10. **[6]** Dunan is taking a calculus course in which the final exam counts for 15% of the total grade. Dunan wishes to have an A in the course, which is defined as a grade of 93% or above. When counting everything but the final exam, he currently has a 92% in the course. What is the minimum integer grade Dunan must get on the final exam in order to get an A in the course?

    Proposed by Samuel Wang

    Solution. $\boxed{99}$

    WLOG, assume that there are a total of 500 points to be gained from this course. Of them, 75 come from the final but 425 come from everything else. Dunan currently has a 92%, corresponding to 391 points. He wishes to have a total of 465 points in the course, meaning he must get at least a 74 on the final exam, or $98.\overline{6}$%. Taking the smallest larger integer, the answer is $\boxed{99}$. □

11. **[6]** Norbert, Eorbert, Sorbert, and Worbert start at the origin of the Cartesian Plane and walk in the positive y, positive x, negative y, and negative x directions respectively at speeds of 1, 2, 3, and 4 units per second respectively. After how many seconds will the quadrilateral with a vertex at each person's location have area 300?

    Proposed by Muztaba Syed

Solution. $\boxed{5}$

If the time is s, then we have a quadrilateral with perpendicular diagonals of length $4s$ and $6s$, which has an area of $\frac{1}{2}\cdot 4s\cdot 6s = 12s^2 = 300 \implies s = \boxed{5}$. □

12. **[6]** Find the sum of the unique prime factors of 1020201.

Proposed by Hannah Shen

Solution. $\boxed{161}$

Note that $1020201 = 1010000 + 10100 + 101$, or $101 * 10101$. 101 is prime, but 10101 can be written as $10^4 + 10^2 + 1 = (10^2+1)^2 - 10^2 = (10^2+10+1)(10^2-10+1)$ through difference of squares. $10^2+10+1 = 111 = 3*37$ and $10^2 - 10 + 1 = 91 = 7*13$, so our sum is $101+37+13+7+3 = \boxed{161}$. □

13. **[6]** HacoobaMatata rewrites the base-10 integers from 0 to 30 inclusive in base 3. How many times does he write the digit 1?

Proposed by Muztaba Syed

Solution. $\boxed{33}$

0 to 26 consists of every string with length 3 consisting of $0,1,2$. By symmetry $\frac{1}{3}$ of the digits will be 1, which gives $27\cdot 3 \div 3 = 27$. A quick check gives $27,28,29,30$ have the digit 1 6 times for an answer of $\boxed{33}$ □

14. **[6]** The fractional part of x is $\frac{1}{7}$. The greatest possible fractional part of x^2 can be written as $\frac{A}{B}$ for relatively prime positive integers A and B. Find $1000A + B$.

Proposed by Hannah Shen

Solution. $\boxed{43049}$

We can express X as $n + \frac{1}{7}$ where n is $\lfloor X \rfloor$. Then, $X^2 = n^2 + \frac{2}{7}n + \frac{1}{49}$. Only the latter two terms can be fractional, so we must maximize the fractional part of $\frac{2}{7}n + \frac{1}{49}$, which occurs when $n \equiv 3 \pmod 7$, making the fractional part $\frac{6}{7} + \frac{1}{49} = \frac{43}{49}$. Hence, $A + B = \boxed{92}$ □

15. **[6]** For how many integers x is $-2x^2 + 8 \geq x^2 - 3x + 2$?

Proposed by bn

Solution. $\boxed{4}$

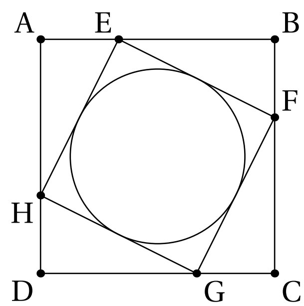
The 2 quadratics intersect at $(-1,6)$ and $(2,0)$. There are $\boxed{4}$ values, $-1,0,1,2$, that satisfy the inequality. □

16. **[6]** In the figure below, circle ω is inscribed in square $EFGH$, which is inscribed in unit square $ABCD$ such that $\overline{EB} = 2\overline{AE}$. If the minimum distance from a point on ω to $ABCD$ can be written as $\frac{P-\sqrt{Q}}{R}$ with Q square-free, find $10000P + 100Q + R$.

Proposed by Hannah Shen

Solution. $\boxed{30506}$

The distance from $ABCD$ to the center of ω is $\frac{1}{2}$; we need to subtract the radius of ω from this value. The radius is half of $\overline{EF}$, or $\frac{1}{2} \cdot \frac{\sqrt{5}}{3}$. So, the distance from $ABCD$ to ω is $\frac{1}{2} - \frac{\sqrt{5}}{6} = \frac{3-\sqrt{5}}{6}$, and $10000A + 100B + C = \boxed{30506}$. □



17. **[6]** There are two base number systems in use in the LHS Math Team. One member writes "13 people use my base, while 23 people use the other, base 12." Another member writes "out of the 34 people in the club, 10 use both bases while 9 use neither." Find the sum of all possible numbers of Math Team members, as a regular decimal number.

Proposed by Jerry Xu

Solution. $\boxed{37}$

Say the first person is speaking in base b. He thus says that $b+3$ people use his base and $2b+3$ people use base $b+2$. The other member could either be speaking in base b or in base $b+2$. If he's speaking in base b, he says there are $3b+4$ people, b of which can speak both bases and 9 of which can speak neither. The Principle of Inclusion and Exclusion tells us that

$$3b+4 = b+3+2b+3-b+9 = 2b+15.$$

In this case, we have $b = 11$, and there are a total of $3 \cdot 11 + 4 = 37$ member in the club in this case. The other case is that the other person speaks in base $b+2$. In this case, he's saying that $3(b+2)+4 = 3b+10$ people are in the club, $b+2$ people speak both bases and 9 people can speak in neither. Once again by the Principle of Inclusion and Exclusion, we have that

$$3b+10 = b+3+2b+3-b-2+9 = 2b+13.$$

We thus have that $b = 3$, but wait! 7 is not a valid numeral in base 3, so this solution is extraneous. Our answer is thus $\boxed{37}$. □

18. **[6]** Sam is taking a test with 100 problems. On this test the questions gradually get harder in such a way that for question i, Sam has a $\frac{(101-i)^2}{100}$% chance to get the question correct. Suppose the expected number of questions Sam gets correct can be written as $\frac{A}{B}$ for relatively prime positive integers A and B. Find $1000A + B$.

Proposed by Samuel Wang

Solution. $\boxed{6767200}$

The expected number is $\frac{100^2+99^2+\cdots+1^2}{10000} = \frac{\frac{100\cdot101\cdot201}{6}}{10000} = \frac{6767}{200}$, so the answer is $\boxed{6767200}$. □

19. **[6]** In an ordered 25-tuple, each component is an integer chosen uniformly and randomly from $\{1,2,3,4,5\}$. Ephram and Zach both copy this tuple into a 5×5 grid, both starting from the top-left corner. Ephram writes five components from left to right to fill one row before continuing down to the next row. Zach writes five components from top to bottom to fill one column before continuing right to the next column. Find the expected number of spaces on their grids where Zach and Ephram have the same integer written.

Proposed by Hannah Shen

Solution. $\boxed{9}$

Since they're copying from the same list, their 5 entries along the diagonal must be the same. For the remaining 20 spaces, there's a $\frac{1}{5}$ chance they have the same number written. So, our expected value is $5+20\cdot\frac{1}{4}=\boxed{9}$. □

20. **[6]** In $\triangle ABC$ with circumcenter O and circumradius 8, $BC=10$. Let r be the radius of the circle that passes through O and is tangent to BC at C. The value of r^2 can be written as $\frac{m}{n}$ for relatively prime positive integers m and n. Find $1000m+n$.

Proposed by Muztaba Syed

Solution. $\boxed{1024039}$

Let the midpoint of BC be M, the circumcenter of ABC be O, the midpoint of OC be D, and the center of the new circle be E. E lies on the perpendicular bisector of OA and $EC\perp BC$. We get that: $\angle ECD=90-\angle OCM=\angle MOC$. Thus $\triangle CMO\sim\triangle EDC$. By the Pythagorean Theorem $OM=\sqrt{39}$, so we have: $\frac{OM}{CM}=\frac{DC}{EC}=\frac{\sqrt{39}}{8}=\frac{4}{EC}\implies EC=\frac{32}{\sqrt{39}}\implies 1000m+n=1024+39=\boxed{1024039}$ □

21. **[6]** Find the number of integer values of n between 1 and 100 inclusive such that the sum of the positive divisors of $2n$ is at least 220% of the sum of the divisors of n.

Proposed by Muztaba Syed

Solution. $\boxed{75}$

Let $v_2(n)$ denote the greatest positive integer k with $2^k\mid n$. If $v_2(n)=0$, then the sum of the divisors will be 3 times the sum of the divisors of n. $v_2(n)=1$ also works, as the sum of the divisors will be $\frac{7}{3}$ times more, which is $>220\%$. But $v_2(n)=3$ won't work, as the sum of the divisors will be $\frac{15}{7}<220\%$. Clearly bigger powers of 2 will only be worse. So multiples of 4 will not work, giving us an answer of $\boxed{75}$ □

22. **[6]** Twenty urns containing one ball each are arranged in a circle. Ernie then moves each ball either 1, 2 or 3 urns clockwise, chosen independently, uniformly, and randomly. The expected number of empty urns after this process is complete can be expressed as $\frac{A}{B}$ for relatively prime positive integers A and B. Find $1000A+B$.

Proposed by Hannah Shen

Solution. $\boxed{160027}$

Label the urns $U_1,U_2,U_3...U_{20}$, and the balls $B_1,B_2,B_3...B_{20}$. For urn U_n to become empty, balls B_{n-1},B_{n-2},B_{n-3} must not end up in U_n. This occurs with probability $\frac{2}{3}*\frac{2}{3}*\frac{2}{3}=\frac{8}{27}$. Hence, the expected number of empty urns is $20*\frac{8}{27}=\frac{160}{27}$, so $1000A+B=\boxed{160027}$. □

23. **[6]** Hannah the cat begins at 0 on a number line. Every second, Hannah jumps 1 unit in the positive or negative direction, chosen uniformly at random. After 7 seconds, Hannah‘s expected distance from 0, in units, can be expressed as $\frac{A}{B}$ for relatively prime positive integers A and B. Find $1000A+B$.

Proposed by Hannah Shen

Solution. $\boxed{5001}$

Note that if the cat is at 0, its expected distance from 0 must increase by 1 in the next second, but at any other number, its expected distance will not change.

We must thus find the probability that the cat is at 0 after each second; this is the value by which the cat's expected distance will increase in the next second.

After n seconds, if n is odd, the probability is clearly 0. If even, there are $\binom{n}{n/2}$ sequences of steps that land the cat at 0, since it must have taken an equal number of left and right steps. There are a total of 2^n sequences it could've taken. Hence, the probability is $\frac{\binom{n}{n/2}}{2^n}$.

Adding 1 to this expression for $n=2,4,6$ yields $1+\frac{1}{2}+\frac{6}{16}+\frac{20}{64}=\frac{35}{16}$, so $1000A+B=\boxed{5001}$. □

24. **[6]** Find the product of all primes $p < 30$ for which there exists an integer n such that p divides $n+(n+1)^{-1} \pmod p$.
Proposed by Muztaba Syed

Solution. $\boxed{5187}$
$n+(n+1)^{-1} \equiv 0 \implies (n+1)^{-1} \equiv -n \implies (n+1)(-n) \equiv 1 \implies n^2+n+1 \equiv 0 \mod p$. If $n \equiv 1 \mod p$, then $p = 3$ works, otherwise: $n^2+n+1 \equiv \frac{n^3-1}{n-1} \equiv 0 \implies n^3 \equiv 1 \mod p$. Because $n \not\equiv 1 \mod p$, the order of $n \mod p$ is 3. This means that there is a solution if and only if $3|p-1$, so $p \equiv 1 \mod 3$, so p can be $7, 13, 19$, so the sum is $3*7*13*19 = \boxed{5187}$. □

25. **[6]** In quadrilateral $ABCD$, $\angle ABD = \angle CBD = \angle CAD$, $AB = 9$, $BC = 6$, and $AC = 10$. The area of $ABCD$ can be expressed as $\frac{P\sqrt{Q}}{R}$ with Q squarefree and P and R relatively prime. Find $10000P + 100Q + R$.
Proposed by Jerry Xu

Solution. $\boxed{134504}$
Say that AC and BD intersect at E. By angle bisector theorem,

$$AE = \frac{9}{9+6} \cdot 10 = 6,$$
$$CE = \frac{6}{9+6} \cdot 10 = 4.$$

Then by Stewarts,

$$9^2 \cdot 4 + 6^2 \cdot 6 = 4 \cdot 6 \cdot 10 + 10 \cdot BE^2$$
$$324 + 216 = 240 + 10 \cdot BE^2$$
$$300 = 10 \cdot BE^2$$
$$BE = \sqrt{30}.$$

We can then note that as $\angle CAD = \angle CBD$, $ABCD$ is cyclic (because the two angles would subtend the same arc on the circle and therefore have equal measure). Then by Power of a Point,

$$6 \cdot 4 = BE \cdot DE$$
$$24 = \sqrt{30} \cdot DE$$
$$DE = \frac{24}{\sqrt{30}} = \frac{24\sqrt{30}}{30} = \frac{4\sqrt{30}}{5}.$$

Now we can note two things: firstly, as $\angle ABD = \angle CBD$, the minor arcs subtended by AD and CD have equal length so $AD = CD$. The second thing we can note is that, by Ptolemy's,

$$6 \cdot AD + 9 \cdot CD = \left(\frac{4\sqrt{30}}{5} + \sqrt{30}\right)(6+4)$$

$$15 \cdot AD = \frac{9\sqrt{30}}{5} \cdot 10$$

$$15 \cdot AD = 18\sqrt{30}$$

$$AD = CD = \frac{6\sqrt{30}}{5}.$$

Now we can simply apply Brahamagupta’s to find that

$$[ABCD] = \sqrt{(s-AB)(s-BC)(s-CD)(s-AD)}$$

$$= \sqrt{\left(\frac{75+12\sqrt{30}}{10}-6\right)\left(\frac{75+12\sqrt{30}}{10}-9\right)\left(\frac{75+12\sqrt{30}}{10}-\frac{6\sqrt{30}}{5}\right)\left(\frac{75+12\sqrt{30}}{10}-\frac{6\sqrt{30}}{5}\right)}$$

$$= \sqrt{\left(\frac{15+12\sqrt{30}}{10}\right)\left(\frac{-15+12\sqrt{30}}{10}\right)\left(\frac{15}{2}\right)^2}$$

$$= \sqrt{(15+12\sqrt{30})(-15+12\sqrt{30})\left(\frac{15}{20}\right)^2}$$

$$= \frac{3}{4}\cdot\sqrt{(12\sqrt{30})^2-15^2}$$

$$= \frac{3}{4}\cdot\sqrt{4095} = \frac{9\sqrt{455}}{4} \implies \boxed{134504}.$$

□