LMT "Fall" Guts Round Solutions- Part 1

Team Name:

1. **[4]** Ephram was born in May 2005. How old will he turn in the first year where the product of the digits of the year number is a nonzero perfect square?

Proposed by Muztab Syed

Solution. $\boxed{107}$

The first such year cannot have any zeroes as digits, and 2111 is the first year after his birthday where that is true. However, product of the digits of this year number is not a nonzero perfect square, but the product of the digits of the next year, 2112, is a perfect square. So, in that year, he turns $2112-2005=\boxed{107}$. □

2. **[4]** Zhao is studying for his upcoming calculus test by reviewing each of the 13 lectures, numbered Lecture 1, Lecture 2, ... Lecture 13. For each n, he spends $5n$ minutes on Lecture n. Which lecture is he reviewing after 4 hours?

Proposed by Aidan Duncan

Solution. $\boxed{10}$

First, note that Zhao spends $\frac{5n(n+1)}{2}$ minutes reviewing the first n lectures. So, we need to find the greatest n such that $\frac{5n(n+1)}{2} < 240$. Simplifying, we get $n(n+1) < 96$, so the desired n is 9. Therefore, after 4 hours, Zhao has finished reviewing the first 9 lectures, so he is reviewing the $\boxed{10}$th lecture. □

3. **[4]** Compute

$$\frac{3^3 \div 3(3)+3}{\frac{3}{3}}+3!.$$

Proposed by Aidan Duncan

Solution. $\boxed{36}$

Computing, we get $3^3 \div 3(3)+3 = 27 \div 3(3)+3 = 9(3)+3 = 30$. Then, since $\frac{3}{3}=1$, we are looking for $30+3! = 30+6 = \boxed{36}$. □

LMT "Fall" Guts Round Solutions- Part 2

Team Name:

4. **[5]** At Ingo's shop, train tickets normally cost \$2, but every 5th ticket costs only \$1. At Emmet's shop, train tickets normally cost \$3, but every 5th ticket is free. Both Ingo and Emmett sold 1000 tickets. Find the absolute difference between their sales, in dollars.

Proposed by Jacob Xu

Solution. $\boxed{600}$

Ingo earns $2*1000-200=1800$, Emmett earns $3*1000-3*200=2400$. Difference is $\boxed{600}$. □

5. **[5]** Ephram paddles his boat in a river with a 4-mph current. Ephram travels at 10 mph in still water. He paddles downstream and then turns around and paddles upstream back to his starting position. Find the proportion of time he spends traveling upstream, as a percentage.

Proposed by Jerry Xu

Solution. $\boxed{70}$

His speed downstream is 14 m/hr and his speed upstream is 6 m/hr. He thus spends 3 hours traveling downstream and 7 hours traveling upstream. Our answer is thus $\frac{7}{10} \longrightarrow \boxed{70}$. □

6. **[5]** The average angle measure of a 13-14-15 triangle is $m°$ and the average angle measure of a 5-6-7 triangle is $n°$. Find $m-n$.

Proposed by Muztaba Syed

Solution. $\boxed{0}$

The average angle measure of any triangle is 60, so the difference is $\boxed{0}$. □

LMT "Fall" Guts Round Solutions- Part 3

Team Name:

7. **[6]** Let $p(x) = x^2 - 10x + 31$. Find the minimum value of $p(p(x))$ over all real x.

Proposed by Muztaba Syed

Solution. $\boxed{7}$

Complete the square to get $p(x) = (x-5)^2 + 6$, so the minimum possible value of $p(x)$ is 6 achieved at $x = 5$. We want $p(x)$ to be as close to 5 as possible, and because $p(x) \geq 6$, 6 is the closest possible value to 5. $p(6) = \boxed{7}$ □

8. **[6]** Michael H. and Michael Y. are playing a game with 4 jellybeans. Michael H starts with 3 of the jellybeans, and Michael Y starts with the remaining 1. Every minute, a Michael flips a coin, and if heads, Michael H takes a jellybean from Michael Y. If tails, Michael Y takes a jellybean from Michael H. Whichever Michael gathers all 4 jellybeans wins. The probability Michael H wins can be written as $\frac{A}{B}$ for relatively prime positive integers A and B. Find $1000A + B$.

Proposed by Michael Yang

Solution. $\boxed{3004}$

In the first minute, there is a probability of $\frac{1}{2}$ that the rope goes to the center, and a probability of $\frac{1}{2}$ that Michael H wins. If it goes to the center, there is an equal chance Michael H and Michael Y will eventually win. Therefore, Michael H wins with probability $\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4} \implies \boxed{3004}$. □

9. **[6]** Define the *digit-product* of a positive integer to be the product of its non-zero digits. Let M denote the greatest five-digit number with a *digit-product* of 360, and let N denote the least five-digit number with a *digit-product* of 360. Find the *digit-product* of $M-N$.

Proposed by Benjamin Yin

Solution. $\boxed{2016}$

To create the largest possible five-digit number M, we will want to place the largest digits at the start of the number. We can notice that 360 = 9*8*5. This means that M=98511 (we place the largest digits in the beginning and fill the rest of the digits with 0's because the 0's don't contribute towards the digit sum). By similar reasoning, to create the smallest possible five-digit number, we place the smallest digits at the start of the number. This means that $N = 10589. M - N = 87922. 8*7*9*2*2 = \boxed{2016}$. □

LMT "Fall" Guts Round Solutions- Part 4

Team Name: ____________

____________ 10. **[7]** Hannah is attending one of the three IdeaMath classes running at LHS, while Alex decides to randomly visit some combination of classes. He won't visit all three classes, but he's equally likely to visit any other combination. The probability Alex visits Hannah's class can be expressed as $\frac{A}{B}$ for relatively prime positive integers A and B. Find $1000A+B$.

Proposed by Jerry Xu

Solution. $\boxed{3007}$

There is a total of $2^3-1=7$ total ways (8 total, and minus 1 for sieging all three) for James to siege Michael's cities. If James is sieging one city, there is only one way for that city to be Ping: if James is sieging two cities, then there are two ways for one of the cities to be Ping (choose which one of the other two is also being sieged). If James is sieging no cities, then there is no way for James to siege Ping. Therefore, our answer is $\frac{3}{7} \implies \boxed{3007}$. □

____________ 11. **[7]** In rectangle $ABCD$, let E be the intersection of diagonal AC and the circle centered at A passing through D. Angle $\angle ACD = 24°$. Find the measure of $\angle CED$ in degrees.

Proposed by Muztaba Syed

Solution. $\boxed{123}$

Because $\triangle AED$ is isosceles we have: $\angle EDA = \frac{180-66}{2} = 57°$, so $\angle EDC = 23°$, so our answer is $180-(24+23) = \boxed{123}$. □

____________ 12. **[7]** During his IdeaMath class, Zach writes the numbers 2, 3, 4, 5, 6, 7, and 8 on a whiteboard. Every minute, he chooses two numbers a and b from the board, erases them, and writes the number $ab+a+b$ on the board. He repeats this process until there's only one number left. Find the sum of all possible remaining numbers.

Proposed by Samuel Wang

Solution. $\boxed{181439}$

$ab+a+b=(a+1)(b+1)-1$. Thus, at the end, we are left with $3*4*5*6*7*8*9-1=\boxed{181439}$. □

LMT "Fall" Guts Round Solutions- Part 5

Team Name: ____________

____________ 13. **[9]** In isosceles right $\triangle ABC$ with hypotenuse AC, Let A' be the point on the extension of AB past A such that $AA'=1$. Let C' be the point on the extension of BC past vertex C such that $CC'=2$. Given that the difference of the areas of triangle $A'BC'$ and ABC is 10, find the area of ABC.

Proposed by Corey Zhao

Solution. $\boxed{18}$

Let us say $AB=BC=x$. The area of ABC is $\frac{x^2}{2}$, and since $A'B=x+1, C'B=x+2$, the area of $A'BC'$ is $\frac{(x+1)(x+2)}{2}$. We are given that $[A'BC']-[ABC]=10 \Rightarrow (x+1)(x+2)-x^2=20 \Rightarrow 3x+2=20 \Rightarrow x=6$. So, the area of ABC is $\frac{x^2}{2}=\boxed{18}$. □

____________ 14. **[9]** Compute the sum of the greatest and least values of x such that

$$(x^2-4x+4)^2+x^2-4x \le 16.$$

Proposed by Evin Liang

Solution. $\boxed{4}$

The largest value of $y = x^2 - 4x + 4$ satisfies $y^2 + y \le 20$ so the largest value is $y = 4$. Since $y = (x-2)^2$ the largest and smallest values of x are 0 and 4, with sum $\boxed{4}$ □

______ 15. **[9]** Ephram is starting a fan club. At the fan club's first meeting, everyone shakes hands with everyone else exactly once, except for Ephram, who is extremely sociable and shakes hands with everyone else twice. Given that a total of 2015 handshakes took place, how many people attended the club's first meeting?

Proposed by Michael Han

Solution. $\boxed{63}$

Let the number of people excluding Ephram be n. We have

$$\binom{n}{2} + 2n = 2015.$$

Solving gives $n = 62$ so there are $62 + 1 = \boxed{63}$ people. □

LMT "Fall" Guts Round Solutions- Part 6

Team Name:

______ 16. **[11]** Let a be a solution to $x^3 - x + 1 = 0$. Find $a^6 - a^2 + 2a$.

Proposed by Muztaba Syed

Solution. $\boxed{1}$

$x^3 = (x-1) \implies x^6 = x^2 - 2x + 1 \implies x^6 - x^2 + 2x = \boxed{1}$ □

______ 17. **[11]** For a positive integer n, $\phi(n)$ is the number of positive integers less than n that are relatively prime to n. Compute the sum of all n for which $\phi(n) = 24$.

Proposed by Evin Liang

Solution. $\boxed{621}$

If p divides n then $p-1$ divides $\phi(n)$. Therefore, the only possible prime factors of n are 13, 7, 5, 3, and 2. Now let's do cases: The largest prime factor is 13. Then the possibilities are $2^2 \cdot 13, 3 \cdot 13, 2 \cdot 3 \cdot 13$. The largest prime factor is 7. Then we can consider sub cases: if 5 divides n the possibilities are $5 \cdot 7$ and $2 \cdot 5 \cdot 7$, otherwise the possibilities are $2^3 \cdot 7, 2^2 \cdot 3 \cdot 7$. The largest prime factor is 5. 5 can only appear once and then $n = 5m$ where $\phi(m) = 6$. Then m has to be divisible by 3^2 and so the possibilities are $3^2 \cdot 5, 2 \cdot 3^2 \cdot 5$. The largest prime factor is 3. Then since 3 divides $\phi(n)$ but 9 does not, n is of the form $3^2 \cdot 2^k$. Therefore the only possibility is $3^2 \cdot 2^3$.

Therefore, the answer is $52 + 39 + 78 + 35 + 70 + 56 + 84 + 45 + 90 + 72 = \boxed{621}$. □

______ 18. **[11]** Let x be a positive integer such that $x^2 \equiv 57 \pmod{59}$. Find the least possible value of x.

Proposed by Evin Liang

Solution. $\boxed{23}$

Because $59 \equiv 3 \mod 4$, we can raise 57 to the $\frac{59+1}{4} = 15$th power to get one possible value of x. We get $57^{15} \equiv (-2)^{15} = -32768 \equiv 36$. The only other possible solution is $59 - 36 = \boxed{23}$, which is the smaller one.

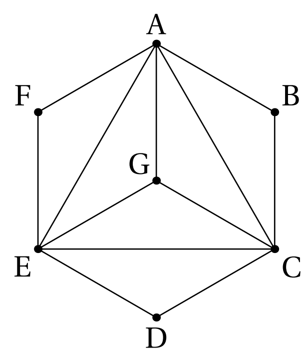
Proof that 23 is the only other solution: if $x^2 \equiv y^2 \mod 59$ for integers x, y such that $0 \le x, y < 59$ and $x \ne y$, we get $(x+y)(x-y) \equiv 0$. Because $x \ne y \implies x - y \not\equiv 0$, we get $x + y \equiv 0$ (because 59 is prime), so $y \equiv -x$ is the only other solution. □

19. **[13]** In the diagram below, find the number of ways to color each vertex red, green, yellow or blue such that no two vertices of a triangle have the same color.

Proposed by Hannah Shen

Solution. $\boxed{192}$

A, E, C and G must be of different colors; there are $4! = 24$ ways to color them. Each of F, B and D can each be one of two colors, since the other two colors are taken by each point's two adjacent vertices. Hence, $24 * 2^3 = \boxed{192}$. □



20. **[13]** In a set with n elements, the sum of the number of ways to choose 3 or 4 elements is a multiple of the sum of the number of ways to choose 1 or 2 elements. Find the number of possible values of n between 4 and 120 inclusive.

Proposed by Muztaba Syed

Solution. $\boxed{38}$

$\binom{n}{3} + \binom{n}{4} = \binom{n+1}{4}$ and $\binom{n}{1} + \binom{n}{2} = \binom{n+1}{2}$ We have that: $\frac{(n+1)(n)(n-1)(n-2)\cdot 2}{(n+1)(n)\cdot 24} = \frac{(n-1)(n-2)}{12}$ is an integer. A quick $\mod 12$ check gives $n \equiv 1, 2, 5, 10 \mod 12$. We see that there are $\boxed{38}$ values that work. □

21. **[13]** In unit square $ABCD$, let Γ be the locus of points P in the interior of $ABCD$ such that $2AP < BP$. The area of Γ can be written as $\dfrac{a\pi + b\sqrt{c}}{d}$ for integers a, b, c, d with c squarefree and $\gcd(a, b, d) = 1$. Find $1000000a + 10000b + 100c + d$.

Proposed by Brandon Ni

Solution. $\boxed{3970354}$

Suppose that A is $(0,0)$, B is $(1,0)$, C is $(1,1)$, D is $(0,1)$ and that point $P = (x, y)$ satisfies $2AP = BP$. Then, write $2AP = BP$ using the Pythagorean Theorem and it eventually comes out to be a circle: $2\sqrt{x^2 + y^2} = \sqrt{y^2 + (1-x)^2}$ $4x^2 + 4y^2 = y^2 + (1-x)^2$ $3x^2 + 3y^2 = 1 - 2x$ $x^2 - \frac{2}{3}x + y^2 = \frac{1}{3}$ Complete the square: $\left(x - \frac{1}{3}\right)^2 + y^2 = \left(\frac{2}{3}\right)^2$ This is the equation of a circle with radius $\frac{2}{3}$ and center $(-\frac{1}{3}, 0)$. Γ is the overlap between the unit square and the circle. Notice that the circle intersects AD at $(0, \frac{\sqrt{3}}{3})$, so the angle between the E, O, and the origin is $60°$. Thus the area of Γ is $\left(\frac{2}{3}\right)^2 \cdot \pi \cdot \frac{1}{6} - \frac{1}{3} \cdot \frac{\sqrt{3}}{3} \cdot \frac{1}{2} = \frac{4\pi - 3\sqrt{3}}{54}$. $1000000a + 10000b + 100c + d = 4000000 - 30000 + 300 + 54 = \boxed{3970354}$ □

____________ 22. **[15]** Ephram, GammaZero, and Orz walk into a bar. Each write some permutation of the letters "LMT" once, then concatenate their permutations one after the other (i.e. LTMTLMTLM would be a possible string, but not LLLMMMTTT). Suppose that the probability that the string "LMT" appears in that order among the new 9-character string can be written as $\frac{A}{B}$ for relatively prime positive integers A and B. Find $1000A+B$.

Proposed by Michael Yang

Solution. $\boxed{67108}$

The number of arrangements of the three original strings such that LMT exists in any one of the three original strings is $36\cdot3-3\cdot6+1=91$ by PIE. Now assume that none of the three original strings have an LMT string. In order to form an LMT string, we can either have one string being TLM and the following string being either TLM or TML or we can have one string being TML or MTL and the following string being MTL. This can take place in either the first and second strings or the second and third strings, so the number of arrangements in which LMT exists in the final string without LMT existing in one of the three original strings is $2\cdot2\cdot2\cdot1\cdot6=48$. Now we must count the overlap, when there are two LMTs formed when concatenating the three strings. This can happen in one of the following five ways: 1. $TMLMTLMTL$, 2. $MTLMTLMTL$, 3. $TLMTMLMTL$, 4. $TLMTLMTLM$, 5.$TLMTLMTML$. So therefore we have $48-5=43$ valid arrangements left. Thus our answer is $\frac{91+43}{216}=\frac{67}{108}\implies \boxed{67108}$. □

____________ 23. **[15]** In $\triangle ABC$ with side lengths $AB=27$, $BC=35$, and $CA=32$, let D be the point at which the incircle is tangent to BC. The value of $\frac{\sin\angle CAD}{\sin\angle BAD}$ can be expressed as $\frac{A}{B}$ for relatively prime positive integers A and B. Find $1000A+B$.

Proposed by Jerry Xu

Solution. $\boxed{9008}$

By the duality principle (essentially just three applications of equal tangents), there exist three positive reals x, y, and z such that $x+y=27$, $x+z=32$, $y+z=35$.Adding up all three equations gives us that $2(x+y+z)=94\to x+y+z=47$. Then we can subtract each of the three original equations to find that $(x,y,z)=(12,15,20)$. By the ratio lemma,

$$\frac{BD}{CD}=\frac{AB}{AC}\cdot\frac{\sin\angle BAD}{\sin\angle CAD}$$
$$\frac{15}{20}=\frac{27}{32}\cdot\frac{\sin\angle BAD}{\sin\angle CAD}$$
$$\frac{\sin\angle BAD}{\sin\angle CAD}=\frac{3}{4}\cdot\frac{32}{27}=\boxed{\frac{8}{9}}.$$

The answer is thus $\boxed{9008}$

Explanation of the duality principle:

Suppose that the above diagram generalizes all triangles ABC, note that $AF=AE$, $BD=BF$, and $CD=CE$ by equal tangents. Denote these values x, y, and z, respectively. We thus have that $x+y=AB, x+z=AC, y+z=BC$. □

____________ 24. **[15]** Let A be the greatest possible area of a square contained in a regular hexagon with side length 1. Let B be the least possible area of a square that contains a regular hexagon with side length 1. The value of $B-A$ can be expressed as $a\sqrt{b}-c$ for positive integers a, b, and c with b squarefree. Find $10000a+100b+c$.

Proposed by Brandon Ni

Solution. $\boxed{70310}$

Note that the largest inscribed square must have all four vertices on the sides of the hexagon and that the square and hexagon share a center. Let s be the side length of the square. We want to express the side length of the hexagon, 1, in terms of s. By dropping two perpendiculars, we can do that. Using a bit of $30-60-90$ triangles, we can find that: $1 = s - 1 + \frac{s\sqrt{3}}{3}$

Solving this gives that $s = 3-\sqrt{3}$ Any rotation of the square will result in shorter diagonals, so this square is the largest. This can be proven by the Pythagorean theorem.

Now, we need to find the area of the smallest square that encloses the hexagon. Again, the square and hexagon share a center, and we can use the symmetry argument to say that the square and hexagon must be symmetric across the square's diagonals. Doing some more Pythagorean Theorem gives that the side length of the square s: $s = \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{2}$

Thus the difference of their areas is $\left(\frac{\sqrt{6}+\sqrt{2}}{2}\right)^2 - (3-\sqrt{3})^2 = 7\sqrt{3}-10$ $10000a+100b+c = \boxed{70310}$ $\square$

LMT "Fall" Guts Round Solutions- Part 9

Team Name:

25. **[10]** Estimate how many days before today this problem was written. If your estimation is E and the actual answer is A, you will receive $\max\left(\left\lfloor 10-\left|\frac{E-A}{2}\right|\right\rfloor, 0\right)$ points.

Proposed by Samuel Wang

Solution. $\boxed{\text{Input Answer Into Spreadsheet}}$ □

26. **[10]** Circle ω_1 is inscribed in unit square $ABCD$. For every integer $1 < n \le 10,000$, ω_n is defined as the largest circle which can be drawn inside $ABCD$ that does not overlap the interior of any of $\omega_1, \omega_2, \dots, \omega_{n-1}$ (If there are multiple such ω_n that can be drawn, one is chosen at random). Let r be the radius of $\omega_{10,000}$. Estimate $\frac{1}{r}$. If your estimation is E and the actual answer is A, you will receive $\max\left(\left\lfloor 10-\left|\frac{E-A}{200}\right|\right\rfloor, 0\right)$ points.

Proposed by Boyan Litchev

Solution. $\boxed{\text{Input Answer Into Spreadsheet}}$ □

27. **[?]** Answer with a positive integer less than or equal to 20. We will compare your response with the response of every other team that answered this problem. When two equal responses are compared, neither team wins. When two unequal responses $A > B$ are compared, A wins if $B \mid A$, and B wins otherwise. If your team wins n times, you will receive $\left\lfloor \frac{n}{2} \right\rfloor$ points.

Proposed by Jeff Lin

Solution. $\boxed{\text{Input Answer Into Spreadsheet}}$ □