3rd Annual Lexington Mathematical Tournament
Team Round

May 5, 2012

1 Potpourri [70]

1. What is 7% of one half of 11% of 20000?

2. Three circles centered at A, B, and C are tangent to each other. Given that $AB = 8$, $AC = 10$, and $BC = 12$, find the radius of circle A.

3. How many positive integer values of x less than 2012 are there such that there exists an integer y for which $\frac{1}{x} + \frac{2}{2y+1} = \frac{1}{y}$?

4. The positive difference between 8 and twice x is equal to 11 more than x. What are all possible values of x?

5. A region in the coordinate plane is bounded by the equations $x = 0$, $x = 6$, $y = 0$, and $y = 8$. A line through $(3, 4)$ with slope 4 cuts the region in half. Another line going through the same point cuts the region into fourths, each with the same area. What is the slope of this line?

6. A polygon is composed of only angles of degrees 138 and 150, with at least one angle of each degree. How many sides does the polygon have?

7. M, A, T, H, and L are all not necessarily distinct digits, with $M \neq 0$ and $L \neq 0$. Given that the sum $MATH + LMT$, where each letter represents a digit, equals 2012, what is the average of all possible values of the three-digit integer LMT?

8. A square with side length $\sqrt{10}$ and two squares with side length $\sqrt{7}$ share the same center. The smaller squares are rotated so that all of their vertices are touching the sides of the larger square at distinct points. What is the distance between two such points that are on the same side of the larger square?

9. Consider the sequence 2012, 12012, 20120, 20121, . . . This sequence is the increasing sequence of all integers that contain "2012". What is the 30th term in this sequence?

10. What is the coefficient of the x^5 term in the simplified expansion of $(x + \sqrt{x} + \sqrt[3]{x})^{10}$?

2 Long Answer Section [130]

Write up full solutions on the provided answer sheets. You are allowed to use the results of earlier problems in the section for later ones, even if you have not solved the earlier problems, but not vice versa.

2.1 The 7 Divisibility Rule

The divisibility rule for 7 is well-known. To check whether a positive integer N is divisible by 7, we cut off (or truncate) the last digit of N and subtract twice that digit from the remaining number. If the result is divisible by 7, then N is too, and if the result is not divisible by 7, then neither is N; we say that the result is divisible by 7 *if and only if* N is divisible by 7. We repeat the truncating and subtracting until our result is small enough that we can instantly tell whether or not it is a multiple of 7. For example, here is the 7 divisibility rule performed on the number 12341: $12341 \to 1234 - 2 \times 1 = 1232 \to 123 - 2 \times 2 = 119 \to 11 - 2 \times 9 = -7$. Since -7 is divisible by 7, so is 12341.

1. [**15**] Let N be a positive integer with at least two digits. Show that if the last digit is truncated, multiplied by 2, and then subtracted from the remaining number, then the result is divisible by 7 if and only if N is divisible by 7.

2. [**25**] Let N be a positive integer with at least two digits. If we truncate the last digit from N and then subtract k times that digit from the remaining number, then the result is divisible by d if and only if N is divisible by d. Find, with proof, the smallest positive integer value possible for k if

   (a) $d = 11$.

   (b) $d = 17$.

2.2 Solving Cubic Equations

Consider the equation $x^3 + 12x = 12$. We will attempt to find a real solution x to this equation.

3. [**10**] Let a and b be real numbers such that $x = a - b$ is a solution to $x^3 + 12x = 12$ and such that $a^3 - b^3 = 12$. What is the value of ab?

4. [**10**] Find a real solution to $x^3 + 12x = 12$ in simplest radical form.

5. [**20**] Consider the equation $x^3 + px = q$ where p and q are constants and p is positive. Find a real solution for x in terms of p and q.

2.3 Tiles and Dominos

Call a *tile* a square with side length 1 and a *domino* a rectangle with dimensions 1×2. In this problem, we will find a formula for the number of ways to fit tiles and dominos into a 1 by n rectangle.

6. [**5**] How many ways are there to fit tiles and dominos into a 1 by n rectangle when

   (a) $n = 3$?

   (b) $n = 4$?

7. [**10**] For $n \geq 1$, let $f(n)$ denote the number of ways to fit tiles and dominos into a 1 by n rectangle. Show that $f(n) = f(n-1) + f(n-2)$ for $n \geq 3$. Hint: Consider two separate cases: 1) The last square of the rectangle is occupied by a tile; 2) The last square of the rectangle is occupied by a domino.

8. [**5**] How many ways are there to fit tiles and dominos into a 1 by 10 rectangle?

9. [**30**] Let $f(n) = c((a+b)^{n+1} - (a-b)^{n+1})$ for positive constants a, b, and c. Find a, b, and c. Hint: Consider $f(1)$, $f(2)$, and $f(3)$.