3rd Annual Lexington Mathematical Tournament - Guts Round Part 1
May 5, 2012

1. [**5**] A \$100 TV has its price increased by 10%. The new price is then decreased by 10%. What is the current price of the TV?

2. [**5**] If $9w + 8x + 7y = 42$ and $w + 2x + 3y = 8$, then what is the value of $100w + 101x + 102y$?

3. [**5**] Find the number of positive factors of $37^3 \cdot 41^3$.

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4. [**5**] Three hoses work together to fill up a pool, and each hose expels water at a constant rate. If it takes the first, second, and third hoses 4, 6, and 12 hours, respectively, to fill up the pool alone, then how long will it take to fill up the pool if all three hoses work together?

5. [**5**] A semicircle has radius 1. A smaller semicircle is inscribed in the larger one such that the two bases are parallel and the arc of the smaller is tangent to the base of the larger. An even smaller semicircle is inscribed in the same manner inside the smaller of the two semicircles, and this procedure continues indefinitely. What is the sum of all of the areas of the semicircles?

6. [**5**] Given that $P(x)$ is a quadratic polynomial with $P(1) = 0$, $P(2) = 0$, and $P(0) = 2012$, find $P(-1)$.

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7. [**6**] Darwin has a paper circle. He labels one point on the circumference as A. He folds A to every point on the circumference on the circle and undoes it. When he folds A to any point P, he makes a blue mark on the point where $\overline{AP}$ and the made crease intersect. If the area of Darwin paper circle is 80, then what is the area of the region surrounded by blue?

8. [**6**] A rectangular wheel of dimensions 6 feet by 8 feet rolls for 28 feet without sliding. What is the total distance traveled by any corner on the rectangle during this roll?

9. [**6**] How many times in a 24-hour period do the minute hand and hour hand of a 12-hour clock form a right angle?

The answers in this section all depend on each other. Find smallest possible solution set.

10. [**6**] Let B be the answer to problem 11. Right triangle ACD has a right angle at C. Squares $ACEF$ and $ADGH$ are drawn such that points D and E do not coincide and points E and H do not coincide. The midpoints of the sides of $ADGH$ are connected to form a smaller square with area B. If the area of $ACEF$ is also B, then find the length CD rounded up to the nearest integer.

11. [**6**] Let C be the answer to problem 12. Find the sum of the digits of C.

12. [**6**] Let A be the answer to problem 10. Given that $a_0 = 1$, $a_1 = 2$, and that $a_n = 3a_{n-1} - a_{n-2}$ for $n \geq 2$, find a_A.

13. [**7**] The expression $\sqrt{2} \times \sqrt[3]{3} \times \sqrt[6]{6}$ can be expressed as a single radical in the form $\sqrt[n]{m}$, where m and n are integers, and n is as small as possible. What is the value of $m + n$?

14. [**7**] Bertie Bott also produces Bertie Bott's Every Flavor Pez. Each package contains 6 peppermint-, 2 kumquat-, 3 chili pepper-, and 5 garlic-flavored candies in a random order. Harold opens a package and slips it into his Dumbledore-shaped Pez dispenser. What is the probability that of the first four candies, at least three are garlic-flavored?

15. [**7**] Quadrilateral $ABCD$ with $AB = BC = 1$ and $CD = DA = 2$ is circumscribed around and inscribed in two different circles. What is the area of the region between these circles?

16. [**7**] Find all values of x that satisfy $\sqrt[3]{x^7} + \sqrt[3]{x^4} = \sqrt[3]{x}$.

17. [**7**] An octagon has vertices at $(2, 1)$, $(1, 2)$, $(-1, 2)$, $(-2, 1)$, $(-2, -1)$, $(-1, -2)$, $(1, -2)$, and $(2, -1)$. What is the minimum total area that must be cut off of the octagon so that the remaining figure is a regular octagon?

18. [**7**] Ron writes a 4 digit number with no zeros. He tells Ronny that when he sums up all the two-digit numbers that are made by taking 2 consecutive digits of the number, he gets 99. He also reveals that his number is divisible by 8. What is the smallest possible number Ron could have written?

19. [**8**] In a certain summer school, 30 kids enjoy geometry, 40 kids enjoy number theory, and 50 kids enjoy algebra. Also, the number of kids who only enjoy geometry is equal to the number of kids who only enjoy number theory and also equal to the number of kids who only enjoy algebra. What is the difference between the maximum and minimum possible numbers of kids who only enjoy geometry and algebra?

20. [**8**] A mouse is trying to run from the origin to a piece of cheese, located at $(4, 6)$, by traveling the shortest path possible along the lattice grid. However, on a lattice point within the region $\{0 \leq x \leq 4, \, 0 \leq y \leq 6, \, (x, y) \neq (0, 0), (4, 6)\}$ lies a rock through which the mouse cannot travel. The number of paths from which the mouse can choose depends on where the rock is placed. What is the difference between the maximum possible number of paths and the minimum possible number of paths available to the mouse?

21. [**8**] The nine points (x, y) with $x, y \in \{-1, 0, 1\}$ are connected with horizontal and vertical segments to their nearest neighbors. Vikas starts at $(0, 1)$ and must travel to $(1, 0)$, $(0, -1)$, and $(-1, 0)$ in any order before returning to $(0, 1)$. However, he cannot travel to the origin 4 times. If he wishes to travel the least distance possible throughout his journey, then how many possible paths can he take?

22. [**8**] Let $g(x) = x^3 - x^2 - 5x + 2$. If a, b, and c are the roots of $g(x)$, then find the value of $((a + b)(b + c)(c + a))^3$.

23. [**8**] A regular octahedron composed of equilateral triangles of side length 1 is contained within a larger tetrahedron such that the four faces of the tetrahedron coincide with four of the octahedron's faces, none of which share an edge. What is the ratio of the volume of the octahedron to the volume of the tetrahedron?

24. [**8**] You are the lone soul at the south-west corner of a square within Elysium. Every turn, you have a $\frac{1}{3}$ chance of remaining at your corner and a $\frac{1}{3}$ chance of moving to each of the two closest corners. What is the probability that after four turns, you will have visited every corner at least once?

25. [**9**] What is the largest integer that cannot be expressed as the sum of nonnegative multiples of 7, 11, and 13?

26. [**9**] Evaluate $12\binom{3}{3} + 11\binom{4}{3} + 10\binom{5}{3} + \cdots + 2\binom{13}{3} + \binom{14}{3}$.

27. [**9**] Worker Bob drives to work at 30 mph half the time and 60 mph half the time. He returns home along the same route at 30 mph half the distance and 60 mph half the distance. What is his average speed along the entire trip, in mph?

28. [**11**] In quadrilateral $ABCD$, diagonals $\overline{AC}$ and $\overline{BD}$ intersect at P with $BP = 4$, $PD = 6$, $AP = 8$, $PC = 3$, and $AB = 6$. What is the length of AD?

29. [**11**] Find all positive integers x such that $x^2 + 17x + 17$ is a square number.

30. [**11**] Zach has ten weighted coins that turn up heads with probabilities $\frac{2}{11^2}$, $\frac{2}{10^2}$, $\frac{2}{9^2}$, ..., $\frac{2}{2^2}$. If he flips all ten coins simultaneously, then what is the probability that he will get an even number of heads?

31. [**13**] Given a sequence $a_1, a_2, \ldots$ such that $a_1 = 3$ and $a_{n+1} = a_n^2 - 2a_n + 2$ for $n \geq 1$, find the remainder when the product $a_1 a_2 \cdots a_{2012}$ is divided by 100.

32. [**13**] Let ABC be an equilateral triangle and let O be its circumcircle. Let D be a point on $\overline{BC}$, and extend $\overline{AD}$ to intersect O at P. If $BP = 5$ and $CP = 4$, then what is the value of DP?

33. [**13**] Surya and Hao take turns playing a game on a calendar. They start with the date January 1 and they can either increase the month to a later month or increase the day to a later day in that month but not both. The first person to adjust the date to December 31 is the winner. If Hao goes first, then what is the first date that he must choose to ensure that he does not lose?

34. [$\leq$ **15**] On May 5, 1868, exactly 144 years before today, Memorial Day in the United States was officially proclaimed. The first Memorial Day took place that year on May 30 at Waterloo, New York. On May 5, 2012, at 12:00 PM, how many results did the search *"memorial day"* on Google return? The search phrase is in quotes, so Google will only return sites that have the words *memorial* and *day* next to each other in that order. Let $N = \max\{0, \lfloor 15.5 \times \frac{\text{Your Answer}}{\text{Actual Answer}} \rfloor\}$. You will earn the number of points equal to $\min\{N, \max\{0, 30 - N\}\}$.

35. [$\leq$ **15**] Estimate the side length of a regular pentagon whose area is 2012. You will earn the number of points equal to $\max\{0, 15 - \lfloor 5 \times |\text{Your Answer} - \text{Actual Answer}| \rfloor\}$.

36. [$\leq$ **15**] Write down one integer between 1 and 15, inclusive. (If you do not, then you will receive 0 points.) Let the number that you submit be x. Let $\bar{x}$ be the arithmetic mean of all of the valid numbers submitted by all of the teams. If $x > \bar{x}$, then you will receive 0 points; otherwise, you will receive x points.