1st Annual Lexington Mathematical Tournament
Team Round

April 3, 2010

1 Potpourri [70]

Transfer the answers only onto the first page of your team's answer packet. Each problem is worth seven points.

1. I open my 2010-page dictionary, whose pages are numbered 1 to 2010 starting on page 1 on the right side of the spine when opened, and ending with page 2010 on the left. If I open to a random page, what is the probability that the two page numbers showing sum to a multiple of 6?

2.
Let A be the number of positive integer factors of 128.
Let B be the sum of the distinct prime factors of 135.
Let C be the units' digit of 3^{81}.
Let D be the number of zeroes at the end of $2^5 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 11^1$.
Let E be the largest prime factor of 999.

Compute $\sqrt[3]{\sqrt{A+B} + \sqrt[3]{D^C + E}}$.

3. The *root mean square* of a set of real numbers is defined to be the square root of the average of the squares of the numbers in the set. Determine the root mean square of 17 and 7.

4. A regular hexagon $ABCDEF$ has area 1. The sides AB, CD, and EF are extended to form a larger polygon with $ABCDEF$ in the interior. Find the area of this larger polygon.

5. For real numbers x, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. For example, $\lfloor 3 \rfloor = 3$ and $\lfloor 5.2 \rfloor = 5$. Evaluate $\lfloor -2.5 \rfloor + \lfloor \sqrt{2} \rfloor + \lfloor -\sqrt{2} \rfloor + \lfloor 2.5 \rfloor$.

6. The mean of five positive integers is 7, the median is 8, and the unique mode is 9. How many possible sets of integers could this describe?

7. How many three digit numbers x are there such that $x+1$ is divisible by 11?

8. Rectangle $ABCD$ is such that $AD = 10$ and $AB > 10$. Semicircles are drawn with diameters AD and BC such that the semicircles lie completely inside rectangle $ABCD$. If the area of the region inside $ABCD$ but outside both semicircles is 100, determine the shortest possible distance between a point X on semicircle AD and Y on semicircle BC.

9. 8 distinct points are in the plane such that five of them lie on a line l, and the other three points lie off the line, in a way such that if some three of the eight points lie on a line, they lie on l. How many triangles can be formed using some three of the 8 points?

10. Carl has 10 Art of Problem Solving books, all exactly the same size, but only 9 spaces in his bookshelf. At the beginning, there are 9 books in his bookshelf, ordered in the following way.

$$A - B - C - D - E - F - G - H - I$$

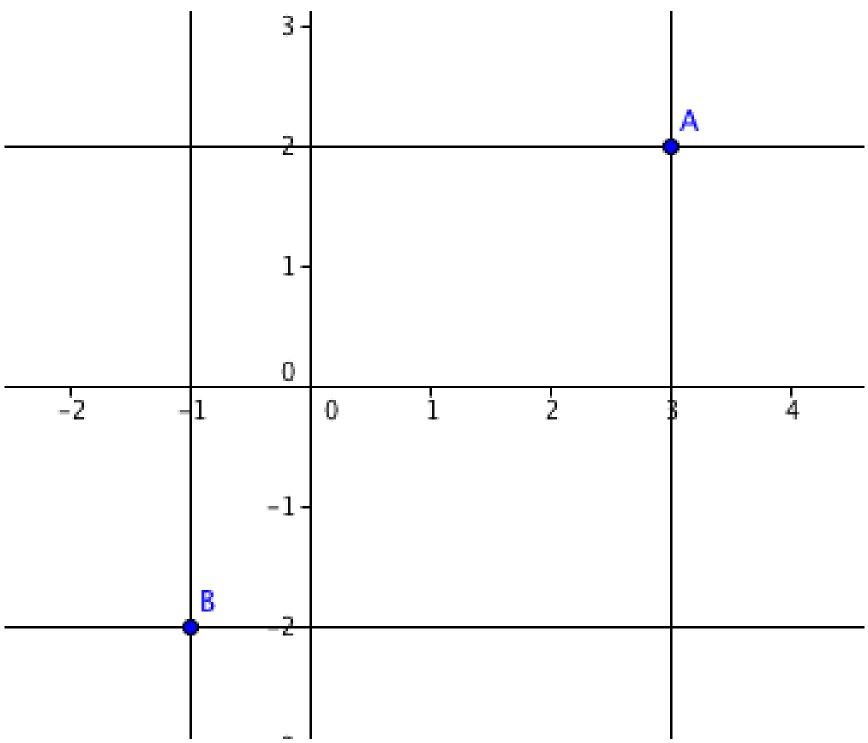
He is holding the tenth book, J, in his hand. He takes the books out one-by-one, replacing each with the book currently in his hand. For example, he could take out A, put J in its place, then take out D, put A in its place, etc. He never takes the same book out twice, and stops once he has taken out the tenth book, which is G. At the end, he is holding G in his hand, and his bookshelf looks like this.

$$C - I - H - J - F - B - E - D - A$$

Give the order (start to finish) in which Carl took out the books, expressed as a 9-letter string (word).

2 The Triangular Lattice [130]

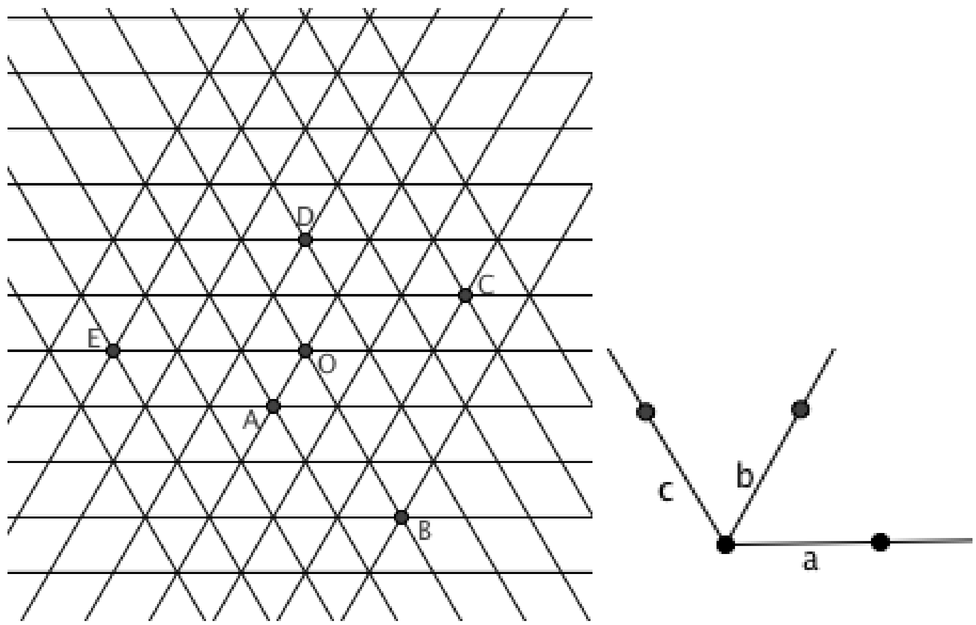
Usually, we work with rectangular coordinates. This means we describe each point's location in the plane by its horizontal and vertical distances from the $x-$ and $y-$ axes. For example, the point $(3, 2)$ is 3 units away from the y-axis and 2 from the x-axis. The point $(-1, -2)$ is 1 unit away from the y-axis, but it is to the left of the y-axis, so we assign the distance a value of -1, and similarly it is at a distance of -2 away from the x-axis.



The rectangular plane (Figure 1)

But here, we will work with instead with the triangular plane, in particular the triangular lattice. To start, we draw lines extending infinitely in three directions: horizontal (parallel to the x-axis), and at angles of $60°$ and $120°$ to the x-axis. It's easy to see that we get a bunch of intersections, whose vertices form a lattice of equilateral triangles. We call each of these vertices a *lattice point*.

We then fix an origin at some lattice point, which we label O. We now want to give each of our lattice points coordinates. We let the point (a, b, c) be the result after moving a units in positive 0° direction (moving to the due east), then b units in the positive 60° direction (moving approximately northeast), and finally c units in the positive 120° direction (moving approximately northwest). As with the rectangular coordinates, we assign negative distances to moving in the opposite directions: for example, moving 1 unit due west gives an a-coordinate of -1.



The triangular plane (Figure 2), and the distances in which we move (Figure 3).

Fact (which you may use without proof): Say we were to take some arbitrary path along the lattice, starting at the origin. Considering the east-west moves, among those say we move a total of a moves east (for example, if our path consists of 2 west and 4 east moves, $a = 2$, and if instead it consists of 2 east and 4 west moves, $a = -2$). Similarly, say we move a total of b moves northeast and c northwest. Then, the end of the path if (a, b, c).

For example, say we took the following path from the origin: E, SE, NE, NW, W, W, E, W, NW. There are 2 east moves and 3 west, so $a = -1$. There is one move northeast and none south-west, so $b = 1$. Finally, there are two moves northwest and one southeast, so $c = 1$. The end of this path is the point $(-1, 1, 1)$.

Write up solutions to the following problems on the provided sheets (keeping in mind that sheets have been designated for problems 2 and 5. Partial credit will be awarded for significant progress that may not constitute a full solution.

1. [10] In Figure 2 (on the previous page), six points are labeled, with O the origin. Match each of the points to the following six triples of coordinates (no justification needed).
(i) $(3, 0, 1)$
(ii) $(2010, -2010, 2010)$
(iii) $(0, 0, -3)$
(iv) $(-4, 5, -3)$
(v) $(3, -3, 0)$
(vi) $(0, -1, 0)$

2. [15] On your answer sheet, drawn is a regular hexagon centered at the origin with side length 4, in the triangular lattice. A pole is placed at each lattice point other than the origin. If Al stands at the origin, he can see the pole at a given lattice point if and only if there isn't another pole directly in front of his view. For example, he can see the point $(0, 1, 1)$, but not $(3, 0, 0)$, since $(1, 0, 0)$ blocks it from Al's view. Neatly circle all of the lattice points where there is a pole that Al can see. How many are there?

3. [20] Start by walking to the point $(a - 1, b, c - 1)$. Now, from here, walk to two points: (a, b, c) and $(a-1, b+1, c-1)$. Conclude that these are the same point, that is, $(a, b, c) = (a-1, b+1, c-1)$.

4. [20] Now, use part 3 repeatedly, to get that $(a, b, c) = (a - c, b + c, 0)$. What happens when c is negative?

5. [20] Using the diagram given on your answer sheet, find a formula for the rectangular coordinates of the triangular lattice point (a, b, c), which is the same as $(a - c, b + c, 0)$.

6. [20] Using the result from problem 5, find a formula for the distance between two arbitrary points in the triangular lattice, (a, b, c) and (d, e, f).

7. [25] What can you say about the triangle formed by the points $(-1, 4, -2)$, $(-3, 1, 0)$, and $(0, -1, -2)$? Why?