1st Annual Lexington Mathematical Tournament
Individual Round

April 3, 2010

1. Two distinct positive even integers sum to 8. Determine the larger of the two integers.

2. Let points A, B, and C lie on a line such that $AB = 1$, $BC = 1$, and $AC = 2$. Let C_1 be the circle centered at A passing through B, and let C_2 be the circle centered at A passing through C. Find the area of the region outside C_1, but inside C_2.

3. Start with a positive integer. Double it, subtract 4, halve it, then subtract the original integer to get x. What is x?

4. Determine the largest positive integer that is a divisor of all three of $A = 2^{2010} \times 3^{2010}$, $B = 3^{2010} \times 5^{2010}$, and $C = 5^{2010} \times 2^{2010}$.

5. Evaluate $2010^2 - 2009 \cdot 2011$.

6. Al has three red marbles and four blue marbles. He draws two different marbles at the same time. What is the probability that one is red and the other is blue?

7. Let $ABCD$ be a square with $AB = 6$. A point P in the interior is 2 units away from side BC and 3 units away from side CD. What is the distance from P to A?

8. How many members are there of the set $\{-79, -76, -73, \ldots, 98, 101\}$?

9. Let ABC and BCD be equilateral triangles, such that $AB = 1$, and $A \neq D$. Find the area of triangle ABD.

10. How many integers less than 2502 are equal to the square of a prime number?

11. Compute the number of positive integers n less than 100 for which $1 + 2 + \ldots + n$ is not divisible by n.

12. Tim is thinking of a positive integer between 2 and 15, inclusive, and Ted is trying to guess the integer. Tim tells Ted how many factors his integer has, and Ted is then able to be certain of what Tim's integer is. What is Tim's integer?

13. Let ABC be a non-degenerate triangle inscribed in a circle, such that AB is the diameter of the circle. Let the angle bisectors of the angles at A and B meet at P. Determine the maximum possible value of $\angle APB$, in degrees.

14. On the team round, an LMT team of six students wishes to divide itself into two distinct groups of three, one group to work on part 1, and one group to work on part 2. In addition, a captain of each group is designated. In how many ways can this be done?

15. Let x and y be real numbers such that $x^2 + y^2 - 22x - 16y + 113 = 0$. Determine the smallest possible value of x.

16. Determine the number of three digit integers that are equal to 19 times the sum of its digits.

17. Al wishes to label the faces of his cube with the integers 2009, 2010, and 2011, with one integer per face, such that adjacent faces (faces that share an edge) have integers that differ by at most 1. Determine the number of distinct ways in which he can label the cube, given that two configurations that can be rotated on to each other are considered the same, and that we disregard the orientation in which each number is written on to the cube.

18. Let l be a line and A be a point such that A is not on l. Let P be a point on l such that segment AP and line l form a 60° angle and $AP = 1$. Extend segment AP past P to a point B on the other side of l. Then, let the perpendicular from B to l have foot M, and extend BM past M to C. Finally, extend CP past P to D. Given that $\frac{BP}{AP} = \frac{CM}{BM} = \frac{DP}{CP} = 2$, determine the area of triangle BPD.

19. Two integers are called *relatively prime* if they share no common factors other than 1. Determine the sum of all positive integers less than 162 that are relatively prime to 162.

20. Let $f(x) = x^5 - 3x^4 + 2x^3 + 6x^2 + x - 14 = a(x-1)^5 + b(x-1)^4 + c(x-1)^3 + d(x-1)^2 + e(x-1) + f$, for some real constants a, b, c, d, e, f. Determine the value of $ab + bc + cd + de + ad + be$.